

Non-affine Landau-Ginzburg models and intersection cohomology

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Abstract

We study Landau-Ginzburg models for numerically effective complete intersections in toric manifolds. These mirror models are partial compactifications of families of Laurent polynomials. We show a mirror statement saying that the quantum \mathcal{D} -module of the ambient part of the cohomology of the submanifold is isomorphic to a certain intersection cohomology \mathcal{D} -module and we deduce Hodge properties of these differential systems.

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1 Introduction

The aim of this paper is the construction of a mirror model for complete intersections in smooth toric varieties. We consider the case where these subvarieties have a numerically effective anticanonical bundle. This includes in particular toric Fano manifolds, whose mirror is usually described by oscillating integrals defined by a family of Laurent polynomials and also the most prominent and classical example of mirror symmetry, namely, that of Calabi-Yau hypersurfaces in toric Fano manifolds. Here the mirror is a family of Calabi-Yau manifolds and the mirror correspondence involves the variation of Hodge structures defined by this family. One interesting feature of our results is that these apparently rather different

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situations occur as special cases of general mirror construction, called *non-affine Landau-Ginzburg model*.

It is well-known that quantum cohomology theories admit expressions in terms of certain differential systems, called quantum \mathcal{D} -modules. This yields a convenient framework in which mirror symmetry is stated as an equivalence of such systems. Moreover, Hodge theoretic aspects of mirror correspondences can be incorporated using the machinery of (mixed) Hodge modules. However, quantum \mathcal{D} -modules have usually irregular singularities, except in the Calabi-Yau case. In our mirror construction, this is taken into account by letting act the Fourier-Laplace functor on various regular \mathcal{D} -modules obtained from the non-affine Landau-Ginzburg model.

The quantum cohomology of a smooth complete intersection (which in our case is given as the zero locus of a generic section of a vector bundle) can be computed using the so-called *Euler-twisted Gromov-Witten invariants*. Basically, these are integrals over moduli spaces of stable maps of pull-backs of cohomology classes on the variety **and** of the Euler class of the vector bundle. It was recently shown in [MM11] that the *ambient part* of the quantum cohomology of the subvariety (consisting of those classes which are induced from cohomology classes of the ambient variety), is given as a quotient of the Euler-twisted quantum cohomology.

From the combinatorial toric data of this vector bundle, we construct in a rather straightforward manner an *affine Landau-Ginzburg model*, which is a family of Laurent polynomials. Then we show that the *twisted quantum \mathcal{D} -module* (which encodes the twisted quantum cohomology) is obtained as the Fourier-Laplace transformation of the **proper** Gauß-Manin system (i.e. measuring cohomology with compact support) of this affine Landau-Ginzburg model. The actual non-affine Landau Ginzburg model is constructed by a certain partial compactification of the affine one, which yields a family of projective varieties. Our main result (Theorem 6.11) states that the ambient *quantum \mathcal{D} -module* is isomorphic to a Fourier-Laplace transformation of the direct image of the intersection cohomology \mathcal{D} -module of the total space of this family, notice that this total space is usually not smooth.

One of the big advantages of using this singular variety together with the intersection cohomology \mathcal{D} -module is the fact that we do not need any kind of resolutions. In particular, we do not need to construct (or suppose the existence of) crepant resolutions like in [Bat94]. Notice also that the recent paper [Iri11] discusses Landau-Ginzburg models of a more special class of subvarieties in toric orbifolds (the so-called nef partitions). In that paper, a mirror statement is shown in terms of A- resp. B-periods, but this construction needs a hypothesis on smoothness of a certain complete intersection (given as the intersection of fibres of several Laurent polynomials, see section 5.2 of loc.cit.).

We will show that the direct image of the intersection cohomology \mathcal{D} -module of the total space is itself (modulo some irrelevant free \mathcal{O} -modules) an intersection cohomology \mathcal{D} -module with respect to a local system measuring the intersection cohomology of the fibers of the projective family. One of the main points in our paper is that this intersection cohomology \mathcal{D} -module admits a hypergeometric description, that is, it can be derived from so-called GKZ-systems (as defined and studied by Gelfand', Kapranov and Zelevinsky). This result is interesting in its own, as in general there are only very few cases where geometrically interesting intersection cohomology \mathcal{D} -modules have an explicit description by differential operators.

Notice that the intersection cohomology \mathcal{D} -module mentioned above underlies a pure Hodge module. From this we can deduce a Hodge-type property of the reduced quantum \mathcal{D} -module (see Corollary 6.12). As already mentioned above, it cannot be itself a Hodge module, as in general it acquires irregular singularities (this never happens for variation of Hodge structures resp. Hodge modules due to Schmid's theorem). Rather, it is part of a non-commutative Hodge (ncHodge) structure due to a key result by Sabbah ([Sab08]).

Let us give a short overview on the paper: In section 2 we discuss generic families of Laurent polynomials defined by an integer matrix with maximal rank. We also define a natural partial compactification of such a family. As mentioned above, it is a projective morphism from a singular space. The construction of this space is rather canonical, it is obtained as the family of hyperplane sections of a projective variety which is toric in a generalized sense, i.e., not necessarily normal. We study the Gauß-Manin system of the uncompactified family, which admits a hypergeometric description due to a central result from [Rei12]. From this we can deduce that the direct image of the intersection \mathcal{D} -module of the space of hyperplane sections is isomorphic to the image of a morphism between two GKZ-systems.

Section 3 introduces the partial localized Fourier transformation. As a consequence of the results of sec-

tion 2, we can identify the Fourier transformed Gauß-Manin and GKZ-systems. Similarly, we describe the Fourier transformation of the intersection cohomology \mathcal{D} -module derived from the compactified family. Then we study natural lattices in these Fourier transformed modules, and show some finiteness properties. Here is substantial difference to our earlier paper [RS10], as we have to study a certain intermediate compactification of the family of Laurent polynomials which is defined on the spectrum of the toric ring defined by the columns of the initial matrix. This variety has a logarithmic structure (in the sense of log geometry), and the good lattice is given by a log twisted de Rham complex. Nevertheless, it still has a nice description using hypergeometric equations, and this allows us to obtain the necessary finiteness result, when we restrict our families to a Zariski open set of the parameter space on which the fibres have sufficiently good properties at infinity. More precisely, we require that the only possible extra singularities at infinity are those already contained in the intermediate compactification given by the (extension of the family to the) toric ring. This is also different to the situation in [RS10] where we had to exclude any singularity at infinity. Let us notice that the assumptions in sections 2 and 3 are rather weak, in particular, we do not suppose that the initial matrix is defined by a toric variety (and, in particular, there is no nefness assumption here). Hence these results may also serve for further studies on Landau-Ginzburg models for not necessarily nef varieties (see, e.g., [GKR12]).

Section 4 is a reminder on notions from quantum cohomology for complete intersections in smooth projective varieties and of the combinatorial description of them for the case of a toric ambient variety. Although most of the material of this section can be found in the literature (e.g., in [MM11]) we include it for the convenience of the reader. Heuristically, the situation of sections 2 and 3 is specialized from here on to the case where the initial matrix (i.e., the one defining the family of Laurent polynomials etc.) is given by the primitive integral generators of the fan of the total space of a bundle over our toric variety. The complete intersection subvariety we are interested in then appears as the zero locus of a generic section of the dual of that bundle.

Section 5 starts with a purely combinatorial result about the semigroups occurring in the situation described in section 4. We show that the corresponding semigroup ring is normal and Gorenstein. The proof here is considerably more involved than that of a related result in [RS10], due to the non-compactness of the underlying toric variety (which is the total space of the bundle alluded to above). By using the machinery of Euler-Koszul complexes, we obtain a duality result for the GKZ-systems associated to the initial matrix, and we also consider a filtered version of this duality theorem.

In section 6, we describe the actual Landau-Ginzburg model of the complete intersection variety. As mentioned above, there are two versions, the *affine* and the *non-affine* one. Both are obtained from the generic families considered in sections 2 and 3 by restricting the parameters to the complexified Kähler moduli space of the ambient toric variety. The concrete description of the above mentioned intersection cohomology \mathcal{D} -module as an image of a morphism between GKZ-systems allows us to identify it with the hypergeometric description of the reduced quantum- \mathcal{D} -module from [MM11]. This yields a geometric explanation of the nature of the *Quot*-ideal appearing in loc.cit., section 4. As our main result (Theorem 6.11), we obtain two mirror statements which says that the affine Landau-Ginzburg model reconstructs the twisted quantum \mathcal{D} -module whereas the non-affine one gives back (via the intersection cohomology \mathcal{D} -module) the reduced quantum \mathcal{D} -module, that is, the ambient part of the quantum cohomology of the complete intersection subvariety. We next discuss the Hodge properties of the reduced quantum \mathcal{D} -module (corollary 6.12 and conjecture 6.13).

Finally, we propose a mirror correspondence (conjecture 6.14) for a slightly different twisted quantum \mathcal{D} -module, which corresponds to the so-called *local Gromov-Witten invariants* of the dual vector bundle. It relies on the forthcoming article [IMM12] where this quantum \mathcal{D} -modules is discussed. Hence we postpone the details of the proof of this conjecture to a subsequent paper.

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2 Intersection Cohomology of Lefschetz fibrations

In this section we use the comparison result between Gauß-Manin systems of Laurent polynomials and GKZ-systems from [Rei12] to describe the direct image of the intersection complex of a natural compactification of a generic family of Laurent polynomials. The input data is an integer matrix B of maximal

rank and the GKZ-system in question will be defined by a certain homogenized matrix \tilde{B} . The main tool is the Radon transformation resp. the Fourier transformation for monodromic \mathcal{D} -modules ([Bry86]).

We start by a short reminder on some basic notions from the theory of algebraic \mathcal{D} -modules. Then we discuss Gauß-Manin systems, GKZ-systems and intersection cohomology \mathcal{D} -modules associated to the above mentioned families. Finally, we show using some facts about *quasi-equivariant* \mathcal{D} -modules that most of the objects considered here behave well with respect to a natural torus action on the parameter space of the families of Laurent polynomials resp. of their compactification.

2.1 Preliminaries

We review very briefly some basic results from the theory of algebraic \mathcal{D} -modules, which will be needed later. Let \mathcal{X} be a smooth algebraic variety (we only consider algebraic varieties defined over \mathbb{C} in the paper) of dimension n and $\mathcal{D}_{\mathcal{X}}$ be the sheaf of algebraic differential operators on \mathcal{X} . We denote by $M(\mathcal{D}_{\mathcal{X}})$ the abelian category of algebraic $\mathcal{D}_{\mathcal{X}}$ -modules on \mathcal{X} and the abelian subcategory of (regular) holonomic $\mathcal{D}_{\mathcal{X}}$ -modules by $M_h(\mathcal{D}_{\mathcal{X}})$ (resp. $(M_{rh}(\mathcal{D}_{\mathcal{X}}))$). The full triangulated subcategory in $D^b(\mathcal{D}_{\mathcal{X}})$ consisting of objects with (regular) holonomic cohomology is denoted by $D_h^b(\mathcal{D}_{\mathcal{X}})$ (resp. $D_{rh}^b(\mathcal{D}_{\mathcal{X}})$).

Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a map between smooth algebraic varieties. Let $\mathcal{M} \in D^b(\mathcal{D}_{\mathcal{X}})$ and $\mathcal{N} \in D^b(\mathcal{D}_{\mathcal{Y}})$, then we denote by $f_+ \mathcal{M} := Rf_*(\mathcal{D}_{\mathcal{Y} \leftarrow \mathcal{X}} \overset{L}{\otimes} \mathcal{M})$ resp. $f^+ \mathcal{N} := \mathcal{D}_{\mathcal{X} \rightarrow \mathcal{Y}} \overset{L}{\otimes} f^{-1} \mathcal{N}$ the direct resp. inverse image for \mathcal{D} -modules. Notice that the functors f_+, f^+ preserve (regular) holonomicity (see e.g., [HTT08, Theorem 3.2.3]). We denote by $\mathbb{D} : D_h^b(\mathcal{D}_{\mathcal{X}}) \rightarrow (D_h^b(\mathcal{D}_{\mathcal{X}}))^{opp}$ the holonomic duality functor. Recall that for a single holonomic $\mathcal{D}_{\mathcal{X}}$ -module M , the holonomic dual is also a single holonomic $\mathcal{D}_{\mathcal{X}}$ -module ([HTT08, Proposition 3.2.1]) and that holonomic duality preserves regular holonomicity ([HTT08, Theorem 6.1.10]).

For a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ between smooth algebraic varieties we additionally define the functors $f_{\dagger} := \mathbb{D} \circ f_+ \circ \mathbb{D}$ and $f^{\dagger} := \mathbb{D} \circ f^+ \circ \mathbb{D}$.

Let $i : \mathcal{Z} \rightarrow \mathcal{X}$ be a closed embedding of a smooth subvariety of codimension d and $j : \mathcal{U} \rightarrow \mathcal{X}$ be the open embedding of its complement. This gives rise to the following triangles for $\mathcal{M} \in D_{rh}^b(\mathcal{D}_{\mathcal{X}})$

$$i_+ i^+ \mathcal{M}[-d] \longrightarrow \mathcal{M} \longrightarrow j_+ j^+ \mathcal{M} \xrightarrow{+1}, \quad (1)$$

$$j_{\dagger} j^{\dagger} \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow i_{\dagger} i^{\dagger} \mathcal{M}[d] \xrightarrow{+1}. \quad (2)$$

The first triangle is [HTT08, Proposition 1.7.1] and the second triangle follows by dualization. We will often use the following base change theorem.

Theorem 2.1 ([HTT08, Theorem 1.7.3]). *Consider the following cartesian diagram of algebraic varieties*

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{f'} & \mathcal{W} \\ g' \downarrow & & \downarrow g \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

then we have the canonical isomorphism $f^+ g_+[d] \simeq g'_+ f'^+[d']$, where $d := \dim \mathcal{Y} - \dim \mathcal{X}$ and $d' := \dim \mathcal{Z} - \dim \mathcal{W}$.

Remark 2.2. *Notice that by symmetry we have also the canonical isomorphism $g^+ f_+[\tilde{d}] \simeq f'_+ g'^{+}[\tilde{d}']$ with $\tilde{d} := \dim \mathcal{W} - \dim \mathcal{X}$ and $\tilde{d}' := \dim \mathcal{Z} - \dim \mathcal{Y}$. In the former case we say we are doing a base change with respect to f , in the latter case with respect to g .*

Remark 2.3. *Using the duality functor we get isomorphisms:*

$$f^{\dagger} g_{\dagger}[-d] \simeq g'_{\dagger} f'^{\dagger}[-d'] \quad \text{and} \quad g^{\dagger} f_{\dagger}[-\tilde{d}] \simeq f'_{\dagger} g'^{\dagger}[-\tilde{d}'].$$

In the sequel, we will consider Fourier-Laplace transformations of various \mathcal{D} -modules. We give a short reminder on the definition and basic properties of the Fourier-Laplace transformation. Let \mathcal{X} be a smooth algebraic variety, U be a finite-dimensional complex vector space and U' its dual vector space. Denote by \mathcal{E}' the trivial vector bundle $\tau : U' \times \mathcal{X} \rightarrow \mathcal{X}$ and by \mathcal{E} its dual. Write $\text{can} : U \times U' \rightarrow \mathbb{C}$ for the canonical morphism defined by $\text{can}(a, \varphi) := \varphi(a)$. This extends to a function $\text{can} : \mathcal{E} \times \mathcal{E}' \rightarrow \mathbb{C}$.

Definition 2.4. Define $\mathcal{L} := \mathcal{O}_{\mathcal{E}' \times_{\mathcal{X}} \mathcal{E}} e^{-\text{can}}$, this is by definition the free rank one module with differential given by the product rule. Denote by $p_1 : \mathcal{E}' \times_{\mathcal{X}} \mathcal{E} \rightarrow \mathcal{E}'$, $p_2 : \mathcal{E}' \times_{\mathcal{X}} \mathcal{E} \rightarrow \mathcal{E}$ the canonical projections. The Fourier-Laplace transformation is then defined by

$$\text{FL}_{\mathcal{X}}(\mathcal{M}) := p_{2+}(p_1^+ \mathcal{M} \overset{L}{\otimes} \mathcal{L}) \quad \mathcal{M} \in D_h^b(\mathcal{D}_{\mathcal{E}'}).$$

If the base \mathcal{X} is a point we will simply write FL. In general, the Fourier-Laplace transformation does not preserve regular holonomicity. However, it does preserve regular holonomicity for the derived category of complexes of \mathcal{D} -modules the cohomology of which are so-called *monodromic* \mathcal{D} -modules. We will give a short reminder on this notion. Let $\chi : \mathbb{C}^* \times \mathcal{E}' \rightarrow \mathcal{E}'$ be the natural \mathbb{C}^* action on the fiber U' and let θ be a coordinate on \mathbb{C}^* . We denote the push-forward $\chi_*(\theta \partial_{\theta})$ as the Euler vector field \mathfrak{E} .

Definition 2.5. [Bry86] A regular holonomic $\mathcal{D}_{\mathcal{E}'}$ -module \mathcal{M} is called *monodromic*, if the Euler field \mathfrak{E} acts locally finite on $\tau_*(\mathcal{M})$, i.e. for a local section v of $\tau_*(\mathcal{M})$ the set $\mathfrak{E}^n(v)$, ($n \in \mathbb{N}$), generates a finite-dimensional vector space. We denote by $D_{\text{mon}}^b(\mathcal{D}_{\mathcal{E}'})$ the derived category of bounded complexes of $\mathcal{D}_{\mathcal{E}'}$ -modules with regular holonomic and monodromic cohomology.

Theorem 2.6. [Bry86]

1. $\text{FL}_{\mathcal{X}}$ preserves complexes with monodromic cohomology.
2. In $D_{\text{mon}}^b(\mathcal{D}_{\mathcal{E}'})$ we have

$$\text{FL}_{\mathcal{X}} \circ \text{FL}_{\mathcal{X}} \simeq \text{Id} \quad \text{and} \quad \mathbb{D} \circ \text{FL}_{\mathcal{X}} \simeq \text{FL}_{\mathcal{X}} \circ \mathbb{D}.$$

3. $\text{FL}_{\mathcal{X}}$ is t -exact with respect to the natural t -structure on $D_{\text{mon}}^b(\mathcal{D}_{\mathcal{E}'})$ resp. $D_{\text{mon}}^b(\mathcal{D}_{\mathcal{E}})$.

Proof. The above statements are stated in [Bry86] for constructible monodromic complexes. One has to use the Riemann-Hilbert correspondence [Bry86, Proposition 7.12, Theorem 7.24] to translate the statements. So the first statement is Corollaire 6.12, the second statement is Proposition 6.13 and the third is Corollaire 7.23 in [Bry86]. \square

We will make occasionally use of the so-called \mathcal{R} -modules. More precisely, let M be a smooth algebraic variety and consider the product of M with the affine line \mathbb{C}_z where z is a fixed coordinate. Then by definition $\mathcal{R}_{\mathbb{C}_z \times M}$ is the $\mathcal{O}_{\mathbb{C}_z \times M}$ -subalgebra of $\mathcal{D}_{\mathbb{C}_z \times M}$ locally generated by $z^2 \partial_z$ and by $z \partial_{x_1}, \dots, z \partial_{x_n}$ where (x_1, \dots, x_n) are local coordinates on M . Notice that $j_M^* \mathcal{R}_{\mathbb{C}_z \times M} \cong \mathcal{D}_{\mathbb{C}_z^* \times M}$, where $j_M : \mathbb{C}_z^* \times M \hookrightarrow \mathbb{C}_z \times M$ is the canonical open embedding.

We will also consider the $\mathcal{O}_{\mathbb{C}_z \times M}$ -subalgebra \mathcal{R}' of \mathcal{R} which is locally generated by $z \partial_{x_1}, \dots, z \partial_{x_n}$ only. The inclusion $\mathcal{R}' \hookrightarrow \mathcal{R}$ induces a functor from the category of \mathcal{R} -modules to the category of \mathcal{R}' -modules, which we denote by $\text{For}_{z^2 \partial_z}$ (“forgetting the $z^2 \partial_z$ -structure”).

2.2 Gauß-Manin systems, hypergeometric \mathcal{D} -modules and the Radon transformation

In this subsection we adapt some results from [Rei12] to our situation. More precisely, for a given generic family of Laurent polynomials, we describe the canonical morphism between its Gauß-Manin-systems with compact support and its usual Gauß-Manin-systems. This mapping can be expressed as a morphism between the corresponding GKZ-systems. We will use this result in the next subsection to describe certain intersection cohomology modules.

We start by fixing our initial data and by introducing the GKZ-hypergeometric \mathcal{D} -modules. Let B be a $s \times t$ -integer matrix such that the columns of B , which we denote by (b_1, \dots, b_t) , generate \mathbb{Z}^s . Consider the torus $S = (\mathbb{C}^*)^s$ and the $t + 1$ -dimensional vector space V (with coordinates $\lambda_0, \lambda_1, \dots, \lambda_t$) as well as its dual V' (with coordinates $\mu_0, \mu_1, \dots, \mu_t$). Define the map

$$\begin{aligned} g : S &\longrightarrow \mathbb{P}(V'), \\ (y_1, \dots, y_s) &\mapsto (1 : \underline{y}^{b_1}, \dots, \underline{y}^{b_t}), \end{aligned} \tag{3}$$

where $\underline{y}^{\underline{b}_i} := \prod_{k=1}^s y_k^{b_{ki}}$ for $i \in \{1, \dots, t\}$. The condition on the columns of the matrix B ensures that this is an embedding. If we denote the closure of the image of g in $\mathbb{P}(V')$ by X , then X is a (possibly non-normal) toric variety in the sense of [GKZ08, Chapter 5]. So we have the following sequence of maps

$$S \xrightarrow{j} X \xrightarrow{i} \mathbb{P}(V'), \quad (4)$$

where j is an open embedding and i a closed embedding.

We will denote the homogeneous coordinates on $\mathbb{P}(V')$ by $(\mu_0 : \dots : \mu_t)$. Let Q be the convex hull of the elements $\{b_0 = 0, \underline{b}_1, \dots, \underline{b}_t\}$ in \mathbb{R}^s . Then by [GKZ08, Chapter 5, Prop 1.9] the projective variety X has a natural stratification by torus orbits $X^0(\Gamma)$, which are in one-to-one correspondence with faces Γ of the polytope Q . The orbit $X^0(\Gamma)$ is isomorphic to $(\mathbb{C}^*)^{\dim(\Gamma)}$ and is specified inside X by the conditions

$$\mu_i = 0 \quad \text{for all } \underline{b}_i \notin \Gamma, \quad \mu_i \neq 0 \quad \text{for all } \underline{b}_i \in \Gamma. \quad (5)$$

In particular the torus $S \subset X$ is given by the face $\Gamma = Q$, i.e. by the equations $\mu_i \neq 0$ for all $i \in \{0, \dots, t\}$.

To this setup we associate the following \mathcal{D} -modules. Write $W = \mathbb{C}^t$ with coordinates $\lambda_1, \dots, \lambda_t$ so that $V = \mathbb{C}_{\lambda_0} \times W$.

Definition 2.7 ([GKZ90], [Ado94]). *Consider a lattice \mathbb{Z}^s and vectors $\underline{b}_1, \dots, \underline{b}_t \in \mathbb{Z}^s$. Moreover, let $\beta = (\beta_1, \dots, \beta_s) \in \mathbb{C}^s$. Write \mathbb{L} for the module of relations among the columns of B . Define*

$$\mathcal{M}_B^\beta := \mathcal{D}_W / ((\square_{\underline{l}})_{\underline{l} \in \mathbb{L}} + (E_k - \beta_k)_{k=1, \dots, s}),$$

where

$$\begin{aligned} \square_{\underline{l}} &:= \prod_{i: l_i < 0} \partial_{\lambda_i}^{-l_i} - \prod_{i: l_i > 0} \partial_{\lambda_i}^{l_i}, \\ E_k &:= \sum_{i=1}^s b_{ki} \lambda_i \partial_{\lambda_i}, \end{aligned}$$

where b_{ki} is the k -th component of \underline{b}_i . The \mathcal{D}_W -module \mathcal{M}_B^β is called a GKZ-system.

As GKZ-systems are defined on the affine space W , we will often work with the D_W -modules of global sections $M_B^\beta := \Gamma(W, \mathcal{M}_B^\beta)$ rather than with the sheaves themselves, where $D_W = \mathbb{C}[\lambda_1, \dots, \lambda_t] \langle \partial_{\lambda_1}, \dots, \partial_{\lambda_t} \rangle$ is the Weyl algebra.

We will also consider a homogenization of the systems above. Let \tilde{B} be the $(s+1) \times (t+1)$ integer matrix with columns $\tilde{b}_0 := (1, \underline{0}), \tilde{b}_1 := (1, \underline{b}_1), \dots, \tilde{b}_t := (1, \underline{b}_t)$.

Definition 2.8. *Consider the hypergeometric system $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ on $V = \mathbb{C}^{t+1}$ associated to the vectors $\tilde{b}_0, \tilde{b}_1, \dots, \tilde{b}_t \in \mathbb{Z}^{s+1}$ and $\tilde{\beta} \in \mathbb{C}^{s+1}$. More explicitly, $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}} := \mathcal{D}_V / \mathcal{I}$, where \mathcal{I} is the sheaf of left ideals in \mathcal{D}_V defined by*

$$\mathcal{I} := \mathcal{D}_V (\square_{\underline{l}})_{\underline{l} \in \mathbb{L}} + \mathcal{D}_V (E_k - \beta_k)_{k=0, \dots, s},$$

where

$$\begin{aligned} \square_{\underline{l}} &:= \partial_{\lambda_0}^{\bar{l}} \cdot \prod_{i: l_i < 0} \partial_{\lambda_i}^{-l_i} - \prod_{i: l_i > 0} \partial_{\lambda_i}^{l_i} \quad \text{if } \bar{l} \geq 0, \\ \square_{\underline{l}} &:= \prod_{i: l_i < 0} \partial_{\lambda_i}^{-l_i} - \partial_{\lambda_0}^{-\bar{l}} \cdot \prod_{i: l_i > 0} \partial_{\lambda_i}^{l_i} \quad \text{if } \bar{l} < 0, \\ E_k &:= \sum_{i=1}^t b_{ki} \lambda_i \partial_{\lambda_i}, \\ E_0 &:= \sum_{i=0}^t \lambda_i \partial_{\lambda_i}. \end{aligned}$$

Let h be the map given by

$$\begin{aligned} h : T &\longrightarrow V', \\ (y_0, \dots, y_s) &\mapsto (\underline{y}^{\tilde{b}_0}, \dots, \underline{y}^{\tilde{b}_t}) = (y_0, y_0 \underline{y}^{\underline{b}_1}, \dots, y_0 \underline{y}^{\underline{b}_t}), \end{aligned} \quad (6)$$

where $T = \mathbb{C}^* \times S = (\mathbb{C}^*)^{s+1}$. Notice that the restriction of h to $\{1\} \times S$ is exactly the map g from formula (3), when seen as a map to the affine chart $\{\mu_0 = 1\} \subset \mathbb{P}(V')$. We will later also need the closure of the image of h in V' , which we denote by Y . Hence Y is the affine cone over X . As a piece of notation, for any matrix C , we write $\mathbb{N}C$ for the semi-group generated by the columns of C . Then we can consider the semi-group ring $\mathbb{C}[\mathbb{N}\tilde{B}]$, which is naturally \mathbb{Z} -graded due to the first line of the matrix \tilde{B} . Hence we can consider the ordinary spectrum of this ring as well as its projective spectrum, and it is clear that we have $Y = \text{Spec } \mathbb{C}[\mathbb{N}\tilde{B}]$ and $X = \text{Proj } \mathbb{C}[\mathbb{N}\tilde{B}]$.

We will now consider natural D_V -linear maps between GKZ-systems, which will induce a shift of the parameter. Let \tilde{B} be as above and consider the map of monoids

$$\begin{aligned} \rho : \mathbb{N}^{t+1} &\longrightarrow \mathbb{N}\tilde{B} \\ e_i &\mapsto \tilde{b}_i \end{aligned} \tag{7}$$

where the e_i are the standard generators of \mathbb{N}^{t+1} . Let $c \in \mathbb{N}^{t+1}$ and put $\tilde{\gamma} := \rho(c)$. Notice that for every $\tilde{\beta} \in \mathbb{C}^{s+1}$ the morphism

$$\begin{aligned} M_{\tilde{B}}^{\tilde{\beta}} &\longrightarrow M_{\tilde{B}}^{\tilde{\beta}+\tilde{\gamma}} \\ P &\mapsto P \cdot \partial^c \end{aligned}$$

is well-defined. Now let $c_1, c_2 \in \rho^{-1}(\tilde{\gamma})$. Because c_1 and c_2 map to the same image, their difference $c_1 - c_2$ is a relation \underline{l} among the columns of the matrix \tilde{B} , thus $\partial^{c_1} - \partial^{c_2} \in (\square_{\underline{l}})$. This shows that $P \cdot p^{c_1} = P \cdot p^{c_2}$ in $M_{\tilde{B}}^{\tilde{\beta}+\tilde{\gamma}}$. Thus, we are lead to the following definition.

Definition 2.9. *Let \tilde{B} and $\tilde{\beta}$ as above. For every $\tilde{\gamma} \in \mathbb{N}\tilde{B}$ define the morphism*

$$\mathcal{M}_{\tilde{B}}^{\tilde{\beta}} \xrightarrow{\cdot \partial^{\tilde{\gamma}}} \mathcal{M}_{\tilde{B}}^{\tilde{\beta}+\tilde{\gamma}}$$

given by right multiplication with ∂^c for any $c \in \rho^{-1}(\tilde{\gamma})$.

In the next lemma, we establish a relation between a direct image under this morphism h and the GKZ-systems just introduced.

Lemma 2.10. *There exists a $\delta_{\tilde{B}} \in \mathbb{N}\tilde{B}$ such that we have an isomorphism*

$$a : \text{FL}(h_+ \mathcal{O}_T) \xrightarrow{\simeq} \mathcal{M}_{\tilde{B}}^{\tilde{\beta}} \tag{8}$$

for every $\tilde{\beta} \in \delta_{\tilde{B}} + (\mathbb{R}_+ \tilde{B} \cap \mathbb{Z}^{s+1})$. Furthermore, we have a dual isomorphism

$$a^\vee : \text{FL}(h_+ \mathcal{O}_T) \xrightarrow{\simeq} \mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'} \tag{9}$$

for every $\tilde{\beta}' \in (\mathbb{R}_+ \tilde{B})^\circ \cap \mathbb{Z}^{s+1}$. For every $\tilde{\beta}, \tilde{\beta}'$ as above, the diagram below commutes up to a non-zero constant

$$\begin{array}{ccc} \mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'} & \xrightarrow{\cdot \partial^{\tilde{\beta}+\tilde{\beta}'}} & \mathcal{M}_{\tilde{B}}^{\tilde{\beta}} \\ a^\vee \downarrow \simeq & & \simeq \uparrow a \\ \text{FL}(h_+ \mathcal{O}_T) & \longrightarrow & \text{FL}(h_+ \mathcal{O}_T), \end{array}$$

where the lower horizontal morphism is induced by the natural morphism $h_+ \mathcal{O}_T \rightarrow h_+ \mathcal{O}_T$.

Proof. By [SW09, Corollary 3.7] we have the isomorphism $\text{FL}(h_+(\mathcal{O}_T \cdot \underline{y}^{\tilde{\beta}})) \simeq \mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ for every $\tilde{\beta} \notin s\text{Res}(\tilde{B})$ where $s\text{Res}(\tilde{B})$ is the set of so-called strongly resonant parameters ([SW09, Definition 3.4]). Here $\mathcal{O}_T \cdot \underline{y}^{\tilde{\beta}}$ is again the free rank one module with differential given by the product rule. Using [Rei12, Lemma 1.16], which says that there exists an $\delta_{\tilde{B}} \in \mathbb{N}\tilde{B}$ such that $\delta_{\tilde{B}} + (\mathbb{R}_+ \tilde{B} \cap \mathbb{Z}^{s+1}) \cap s\text{Res}(\tilde{B}) = \emptyset$ and the fact that

$\mathcal{O}_T \simeq \mathcal{O}_T \cdot \underline{y}^{\tilde{\gamma}}$ for every $\tilde{\gamma} \in \mathbb{Z}^{s+1}$, the first statement follows. The second statement follows from taking the holonomic dual of (8), namely, we put

$$a^\vee := \mathbb{D}a : \mathbb{D}\mathcal{M}_{\tilde{B}}^{\tilde{\beta}} \xrightarrow{\simeq} \mathbb{D}\mathrm{FL}(h_+\mathcal{O}_T) \simeq \mathrm{FL}(\mathbb{D}h_+\mathcal{O}_T) \simeq \mathrm{FL}(h_+\mathcal{O}_T)$$

and then we conclude by applying [Rei12, Proposition 1.23].

The last statement follows from the fact that the only non-zero morphism between $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}$ and $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ is right multiplication $\partial^{\tilde{\beta}+\tilde{\beta}'}$ up to a non-zero constant (cf. [Rei12, Proposition 1.24]). \square

We will denote by $Z \subset \mathbb{P}(V') \times V$ the universal hyperplane given by $Z := \{\sum_{i=0}^t \lambda_i \mu_i = 0\}$ and by $U := (\mathbb{P}(V') \times V) \setminus Z$ its complement. Consider the following diagram

$$\begin{array}{ccccc} & & U & & \\ & \swarrow \pi_1^U & \downarrow j_U & \searrow \pi_2^U & \\ \mathbb{P}(V') & \xleftarrow{\pi_1} & \mathbb{P}(V') \times V & \xrightarrow{\pi_2} & V \\ & \swarrow \pi_1^Z & \downarrow i_Z & \searrow \pi_2^Z & \\ & & Z & & \end{array}$$

We will use in the sequel several variants of the so-called Radon transformation. These are functors from $D_{rh}^b(\mathcal{D}_{\mathbb{P}(V')})$ to $D_{rh}^b(\mathcal{D}_V)$ given by

$$\begin{aligned} \mathcal{R}(M) &:= \pi_{2+}^Z (\pi_1^Z)^+ M \simeq \pi_{2+} i_{Z+} i_Z^+ \pi_1^+ M, \\ \mathcal{R}^\circ(M) &:= \pi_{2+}^U (\pi_1^U)^+ M \simeq \pi_{2+} j_{U+} j_U^+ \pi_1^+ M, \\ \mathcal{R}_c^\circ(M) &:= \pi_{2+}^U (\pi_1^U)^+ M \simeq \pi_{2+} j_{U\dagger} j_U^+ \pi_1^+ M, \\ \mathcal{R}_{cst}(M) &:= \pi_{2+} (\pi_1)^+ M, \end{aligned}$$

The adjunction triangle corresponding to the open embedding j_U and the closed embedding i_Z gives rise to the following triangles of Radon transformations.

$$\mathcal{R}[-1](M) \longrightarrow \mathcal{R}_{cst}(M) \longrightarrow \mathcal{R}^\circ(M) \xrightarrow{+1}, \quad (10)$$

$$\mathcal{R}_c^\circ(M) \longrightarrow \mathcal{R}_{cst}(M) \longrightarrow \mathcal{R}[1](M) \xrightarrow{+1}, \quad (11)$$

where the second triangle is dual to the first.

We can now introduce the generic family of Laurent polynomials mentioned at the beginning of this subsection. It is defined by the columns of the matrix B , more precisely, we put

$$\begin{aligned} \varphi_B : S \times W &\longrightarrow V = \mathbb{C}_{\lambda_0} \times W, \\ (y_1, \dots, y_s, \lambda_1, \dots, \lambda_t) &\mapsto \left(- \sum_{i=1}^t \lambda_i \underline{y}^{b_i}, \lambda_1, \dots, \lambda_t \right). \end{aligned} \quad (12)$$

The following theorem of [Rei12] constructs a morphism between the Gauß-Manin system $\mathcal{H}^0(\varphi_{B,+}\mathcal{O}_{S \times W})$ resp. the its proper version $\mathcal{H}^0(\varphi_{B,\dagger}\mathcal{O}_{S \times W})$ and certain GKZ-hypergeometric systems. For this we apply the triangle (10) to $M = g_\dagger \mathcal{O}_S$ and the triangle (11) to $M = g_+ \mathcal{O}_S$, which gives us the result.

Theorem 2.11. [Rei12, Lemma 1.16, Theorem 2.7] *There exists an $\delta_{\tilde{B}} \in \mathbb{N}\tilde{B}$ such that for every $\tilde{\beta} \in \delta_{\tilde{B}} + \mathbb{R}_+\tilde{B} \cap \mathbb{Z}^{s+1}$ and every $\tilde{\beta}' \in (\mathbb{N}\tilde{B})^\circ = \mathbb{N}\tilde{B} \cap (\mathbb{R}_+\tilde{B})^\circ$, the following sequences of \mathcal{D}_V -modules*

are exact and dual to each other:

$$\begin{array}{ccccccc}
H^{s-1}(S, \mathbb{C}) \otimes \mathcal{O}_V & \mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W}) & \mathcal{M}_{\tilde{B}}^{\tilde{\beta}} & H^s(S, \mathbb{C}) \otimes \mathcal{O}_V & & & \\
\uparrow \simeq & \uparrow \simeq & \uparrow \simeq & \uparrow \simeq & & & \\
0 \rightarrow \mathcal{H}^{-1}(\mathcal{R}_{cst}(g_+ \mathcal{O}_S)) \rightarrow \mathcal{H}^0(\mathcal{R}(g_+ \mathcal{O}_S)) \rightarrow \mathcal{H}^0(\mathcal{R}_c^\circ(g_+ \mathcal{O}_S)) \rightarrow \mathcal{H}^0(\mathcal{R}_{cst}(g_+ \mathcal{O}_S)) \rightarrow 0 & & & & & & \\
& & & & & & \\
0 \leftarrow \mathcal{H}^1(\mathcal{R}_{cst}(g_+ \mathcal{O}_S)) \leftarrow \mathcal{H}^0(\mathcal{R}(g_+ \mathcal{O}_S)) \leftarrow \mathcal{H}^0(\mathcal{R}_c^\circ(g_+ \mathcal{O}_S)) \leftarrow \mathcal{H}^0(\mathcal{R}_{cst}(g_+ \mathcal{O}_S)) \leftarrow 0 & & & & & & \\
\downarrow \simeq & \downarrow \simeq & \downarrow \simeq & \downarrow \simeq & & & \\
H_c^{s+1}(S, \mathbb{C}) \otimes \mathcal{O}_V & \mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W}) & \mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'} & H_c^s(S, \mathbb{C}) \otimes \mathcal{O}_V & & &
\end{array}$$

If moreover $\mathbb{N}\tilde{B}$ is saturated, then the vector $\delta_{\tilde{B}}$ can be taken to be $\underline{0} \in \mathbb{N}\tilde{B}$, in particular, the above statement holds for $\tilde{\beta} = \underline{0} \in \mathbb{Z}^{s+1}$.

Thus we get the following exact 4-term sequences which can be connected vertically by the map $\eta : H^0(\mathcal{R}(g_+ \mathcal{O}_S)) \rightarrow H^0(\mathcal{R}(g_+ \mathcal{O}_S))$ induced by the natural morphism $g_+ \mathcal{O}_S \rightarrow g_+ \mathcal{O}_S$. Define θ to be the composition $\kappa_2 \circ \eta \circ \kappa_1$. The next result gives a concrete description of this morphism:

$$\begin{array}{ccccccc}
0 \longrightarrow H^{s-1}(S, \mathbb{C}) \otimes \mathcal{O}_V \longrightarrow \mathcal{H}^0(\mathcal{R}(g_+ \mathcal{O}_S)) \xrightarrow{\kappa_2} \mathcal{M}_{\tilde{B}}^{\tilde{\beta}} \longrightarrow H^s(S, \mathbb{C}) \otimes \mathcal{O}_V \longrightarrow 0 & & & & & & \\
& & \uparrow \eta & & \uparrow \theta & & \\
0 \longleftarrow H_c^{s+1}(S, \mathbb{C}) \otimes \mathcal{O}_V \longleftarrow \mathcal{H}^0(\mathcal{R}(g_+ \mathcal{O}_S)) \xleftarrow{\kappa_1} \mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'} \longleftarrow H_c^s(S, \mathbb{C}) \otimes \mathcal{O}_V \longleftarrow 0 & & & & & &
\end{array}$$

Lemma 2.12. *The morphism θ is induced by right multiplication with $\partial^{\tilde{\beta}+\tilde{\beta}'}$ up to a non-zero constant.*

Proof. Once we can prove that $\kappa_2 \circ \eta \circ \kappa_1$ is not equal to zero we apply a rigidity result of [Rei12, Proposition 1.24] which says that the only maps between $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}$ and $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ is right-multiplication with $c \cdot \partial^{\tilde{\beta}+\tilde{\beta}'}$ for $c \in \mathbb{C}$. We only have to show that $\kappa_2 \circ \eta \circ \kappa_1$ becomes an isomorphism after micro-localizing with respect to $\partial^0 \cdots \partial^t$. This is sufficient as the microlocalization of the GKZ-systems $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ resp. $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}$ are not zero for otherwise the sheaves $h_+ \mathcal{O}_T$ and $h_+ \mathcal{O}_T$ would be supported on the divisor $\{\mu_0 \cdot \mu_1 \cdots \mu_t = 0\}$, which is obviously wrong.

It is clear that κ_1 and κ_2 become isomorphisms after (micro-)localization with respect to $\partial_0 \cdots \partial_t$ because these maps have \mathcal{O}_V -free kernel and cokernel. It remains to prove that η is an isomorphism after this micro-localization. To prove this we will use a theorem of [DE03] which compares the Radon transformation with the Fourier-Laplace transformation for \mathcal{D} -modules. Consider the following diagram

$$\begin{array}{ccccc}
T & \xrightarrow{h} & V' & \xleftarrow{p} & Bl_0(V') \\
& \searrow \tilde{h} & \uparrow j_0 & & \swarrow q \\
& & V' \setminus \{0\} & & \\
\pi_T \downarrow & & \downarrow \pi & & \\
S & \xrightarrow{g} & \mathbb{P}(V') & &
\end{array}$$

where $Bl_0(V') \subset \mathbb{P}(V') \times V'$ is the blow-up of 0 in V' and q is the restriction of the projection to the first component. Notice that the map $h : T \rightarrow V'$ from formula (6) factors via $V' \setminus \{0\}$, that is, we have $h = j_0 \circ \tilde{h}$, where $j_0 : V' \setminus \{0\} \hookrightarrow V'$ is the canonical inclusion.

It follows from [DE03, Proposition 1] that we have the following isomorphism

$$\mathcal{R}(g_+ \mathcal{O}_S) \simeq \text{FL}(p_+ q^+ g_+ \mathcal{O}_S) \quad (13)$$

and its holonomic dual

$$\mathcal{R}(g_{\dagger} \mathcal{O}_S) \simeq \text{FL}(p_+ q^+ g_{\dagger} \mathcal{O}_S), \quad (14)$$

where we have used $\mathcal{R} \circ \mathbb{D} = \mathbb{D} \circ \mathcal{R}$, $\text{FL} \circ \mathbb{D} = \mathbb{D} \circ \text{FL}$, $p_+ \circ \mathbb{D} = \mathbb{D} \circ p_+$ (p is proper) and $q^+ \circ \mathbb{D} = \mathbb{D} \circ q^+$ (q is smooth). Recall that we want to show that the morphism

$$\mathcal{H}^0(\mathcal{R}(g_{\dagger} \mathcal{O}_S)) \xrightarrow{\eta} \mathcal{H}^0(\mathcal{R}(g_+ \mathcal{O}_S)),$$

becomes an isomorphism after localization with respect to $\partial_{\lambda_0} \cdots \partial_{\lambda_t}$. Using the isomorphisms (13) and (14) and the fact that FL is an exact functor and that it exchanges the action of μ_i and ∂_{λ_i} we see that it is enough to show that

$$\mathcal{H}^0(p_+ q^+ g_{\dagger} \mathcal{O}_S) \longrightarrow \mathcal{H}^0(p_+ q^+ g_+ \mathcal{O}_S) \quad (15)$$

becomes an isomorphism after localization with respect to $\mu_0 \cdots \mu_t$. In other words, we have to show that the kernel and the cokernel of the morphism (15) are supported on $\{\mu_0 \cdots \mu_t = 0\} \subset V'$. Obviously, we have $\{0\} \subset \{\mu_0 \cdots \mu_t = 0\}$ and hence $V' \setminus \{\mu_0 \cdots \mu_t = 0\} \subset V' \setminus \{0\}$. It is thus sufficient to show that kernel and cokernel of the restriction of the morphism (15) to $V' \setminus \{0\}$ are supported on $\{\mu_0 \cdots \mu_t = 0\} \setminus \{0\}$. Notice that the restriction of $\mathcal{H}^0(p_+ q^+ g_{\dagger} \mathcal{O}_S)$ resp. $\mathcal{H}^0(p_+ q^+ g_+ \mathcal{O}_S)$ to $V' \setminus \{0\}$ is isomorphic to $\mathcal{H}^0(\pi^+ g_{\dagger} \mathcal{O}_S)$ resp. $\mathcal{H}^0(\pi^+ g_+ \mathcal{O}_S)$. Thus the kernel and the cokernel of (15) are supported on $\{\mu_0 \cdots \mu_t = 0\}$ if and only if kernel and cokernel of

$$\mathcal{H}^0(\pi^+ g_{\dagger} \mathcal{O}_S) \longrightarrow \mathcal{H}^0(\pi^+ g_+ \mathcal{O}_S)$$

are supported on $\{\mu_0 \cdots \mu_t = 0\} \setminus \{0\}$. The map π is smooth and therefore π^+ is an exact functor. It is therefore enough to show that kernel and cokernel of

$$\mathcal{H}^0(g_{\dagger} \mathcal{O}_S) \longrightarrow \mathcal{H}^0(g_+ \mathcal{O}_S)$$

are supported on $\{\mu_0 \cdots \mu_t = 0\} \subset \mathbb{P}(V')$. But this follows from the description of the map g , namely, by the remark right after equation (5) the support of the cone of the morphism $g_{\dagger} \mathcal{O}_S \rightarrow g_+ \mathcal{O}_S$ is contained in $\{\mu_0 \cdots \mu_t = 0\}$. \square

2.3 Intersection cohomology \mathcal{D} -modules

As mentioned in the beginning of this section, our aim is to describe a \mathcal{D}_V -module derived from the intersection complex of a natural compactification of the family of Laurent polynomials φ_B as defined in formula (12). This module will actually appear as the Radon transformation of the (\mathcal{D} -module corresponding to the) intersection complex of the variety $X \subset \mathbb{P}(V')$.

We start by fixing some notations concerning these \mathcal{D} -modules. Let \mathcal{P} be a smooth variety and $\mathcal{U} \subset \mathcal{P}$ be a smooth subvariety, write \mathcal{X} for the closure of \mathcal{U} inside \mathcal{P} , $j_{\mathcal{U}} : \mathcal{U} \hookrightarrow \mathcal{X}$ for the open embedding of \mathcal{U} in \mathcal{X} and $i_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{P}$ for the closed embedding of the closure of \mathcal{X} in \mathcal{P} . Consider the abelian category $\text{Perv}(\mathcal{P})$ of perverse sheaves on \mathcal{P} (with respect to middle perversity). For a reference about the definition and basic properties of perverse sheaves, see [Dim04]. Recall that the simple objects in $\text{Perv}(\mathcal{P})$ are the objects $(i_{\mathcal{X}})_! IC(\mathcal{X}, \mathcal{L})$ where \mathcal{L} is an irreducible local system on \mathcal{U} and $IC(\mathcal{X}, \mathcal{L})$ is the intersection complex of \mathcal{X} with coefficient in \mathcal{L} , that is the image of the morphism ${}^p\mathcal{H}^0((j_{\mathcal{U}})_! \mathcal{L}) \rightarrow {}^p\mathcal{H}^0((Rj_{\mathcal{U}})_* \mathcal{L})$ in $\text{Perv}(\mathcal{X})$. We will denote the corresponding \mathcal{D} -module on \mathcal{P} by $\mathcal{M}^{IC}(\mathcal{X}, \mathcal{L})$. If \mathcal{L} is the constant sheaf $\underline{\mathbb{C}}_{\mathcal{U}}$ we will simply write $\mathcal{M}^{IC}(\mathcal{X})$. The p -th intersection cohomology group of \mathcal{X} (see [GM83]) is denoted by $IH^p(\mathcal{X})$ and is obtained from the intersection complex by the formula $IH^p(\mathcal{X}) = \mathbb{H}^{p-\dim(\mathcal{X})}(IC(\mathcal{X}, \mathbb{C}_{\mathcal{U}}))$. We will apply this formalism to the special situation where $\mathcal{U} = g(S)$ (where g is the embedding defined by formula (3)), $\mathcal{X} = X$ and $\mathcal{P} = \mathbb{P}(V')$. The module $\mathcal{M}^{IC}(X)$ is the image of the morphism $g_{\dagger} \mathcal{O}_S \rightarrow g_+ \mathcal{O}_S$. In the next result, we will compute the Radon transformation of this module.

Proposition 2.13. *In the above situation, we have the following (non-canonical) isomorphism of \mathcal{D}_V -modules*

$$\mathcal{H}^0 \mathcal{R}(\mathcal{M}^{IC}(X)) \simeq \mathcal{M}^{IC}(X^\circ, \mathcal{L}) \oplus (IH^{s-1}(X) \otimes \mathcal{O}_V),$$

and

$$\begin{aligned}\mathcal{H}^i \mathcal{R}(\mathcal{M}^{IC}(X)) &\simeq IH^{i+s+1}(X) \otimes \mathcal{O}_V \quad \text{for } i > 0, \\ \mathcal{H}^i \mathcal{R}(\mathcal{M}^{IC}(X)) &\simeq IH^{i+s-1}(X) \otimes \mathcal{O}_V \quad \text{for } i < 0,\end{aligned}$$

where X° is some subvariety of V , \mathcal{L} some local system on some smooth open subset of X° and the \mathcal{C}_i are free \mathcal{O}_V -modules.

Proof. Using the comparison isomorphism between the Radon transformation and the Fourier-Laplace transformation (equation (13)) from above, we have

$$\begin{aligned}\mathcal{H}^i \mathcal{R}(\mathcal{M}^{IC}(X)) &\simeq \mathcal{H}^i \text{FL}(p_+ q^+ \mathcal{M}^{IC}(X)) \\ &\simeq \text{FL} \mathcal{H}^i(p_+ q^+ \mathcal{M}^{IC}(X)) \\ &\simeq \text{FL} \mathcal{H}^i(p_+ \mathcal{M}^{IC}(q^{-1}(X))),\end{aligned}$$

where the second isomorphism follows from the exactness of FL and the last isomorphism follows from the smoothness of q . We now apply the decomposition theorem [Sai88, corollaire 3, equation 0.12] which gives

$$\mathcal{H}^i(p_+ \mathcal{M}^{IC}(q^{-1}(X))) \simeq \bigoplus_k \mathcal{M}^{IC}(Y_k^i, \mathcal{L}_k^i) \quad (16)$$

for some subvarieties $Y_k^i \subset V'$ and some local systems \mathcal{L}_k^i on a Zariski open subset of Y_k . Notice that

$$\begin{aligned}j_0^+ \mathcal{H}^i(p_+ \mathcal{M}^{IC}(q^{-1}(X))) &\simeq j_0^+ \mathcal{H}^i(p_+ q^+ \mathcal{M}^{IC}(X)) \\ &\simeq \mathcal{H}^i(j_0^+ p_+ q^+ \mathcal{M}^{IC}(X)) \\ &\simeq \mathcal{H}^i(\pi^+ \mathcal{M}^{IC}(X)) \\ &\simeq \mathcal{H}^i(\mathcal{M}^{IC}(\pi^{-1}(X))),\end{aligned}$$

which is equal to 0 for $i \neq 0$ and equal to $\mathcal{M}^{IC}(Y \setminus \{0\})$ for $i = 0$ (recall from subsection 2.2, more precisely, from the discussion before Lemma 2.10, that Y is the cone of X in V'). Thus the decomposition from (16) becomes

$$\mathcal{H}^0(p_+ \mathcal{M}^{IC}(q^{-1}(X))) \simeq \mathcal{M}^{IC}(Y) \oplus \mathcal{S}_0,$$

resp.

$$\mathcal{H}^i(p_+ \mathcal{M}^{IC}(q^{-1}(X))) \simeq \mathcal{S}_i \quad i \neq 0,$$

where the \mathcal{S}_i are \mathcal{D} -modules with support at 0, i.e. $\mathcal{S}_i \simeq i_{0+} S_i$, where the S_i are finite-dimensional vector spaces and $i_0 : \{0\} \rightarrow V'$ is the natural embedding. We now use the fact that FL is an equivalence of categories, which means that it transforms simple object to simple objects, so we set

$$\mathcal{M}^{IC}(X^\circ, \mathcal{L}) := \text{FL}(\mathcal{M}^{IC}(Y)). \quad (17)$$

It also transforms \mathcal{D} -modules with support at 0 to free \mathcal{O} -modules, i.e. $\text{FL}(\mathcal{S}_i) \simeq S_i \otimes \mathcal{O}_V$. In order to show the claim, we have to compute the S_i . Recall that we have

$$p_+ q^+ \mathcal{M}^{IC}(X) \simeq \bigoplus_j \mathcal{H}^j(p_+ q^+ \mathcal{M}^{IC}(X))[-j] \simeq \bigoplus_{j \neq 0} \mathcal{S}_j \oplus \mathcal{S}_0 \oplus \mathcal{M}^{IC}(Y), \quad (18)$$

where the first isomorphism is non-canonical. We compute

$$H^i(a_{V'})_+ p_+ (q^+ \mathcal{M}^{IC}(X)) \simeq H^i(a_{\mathbb{P}})_+ q_+ (q^+ \mathcal{M}^{IC}(X)) \simeq H^i(a_{\mathbb{P}})_+ \mathcal{M}^{IC}(X)[1] \simeq IH^{i+s+1}(X)$$

(here $a_{V'} : V' \rightarrow \{pt\}$ resp. $a_{\mathbb{P}} : \mathbb{P}(V') \rightarrow \{pt\}$ are the projections to a point), where the second isomorphism follows from [KS94, Corollary 2.7.7 (iv)] and the Riemann-Hilbert correspondence. For the right hand side of Equation (18) we get

$$\begin{aligned}H^i(a_{V'})_+ \left(\bigoplus_{j \neq 0} \mathcal{S}_j \oplus \mathcal{S}_0 \oplus \mathcal{M}^{IC}(Y) \right) &\simeq S_i \quad \text{for } i \geq 0, \\ H^i(a_{V'})_+ \left(\bigoplus_{j \neq 0} \mathcal{S}_j \oplus \mathcal{S}_0 \oplus \mathcal{M}^{IC}(Y) \right) &\simeq S_i \oplus IH^{i+s+1}(Y) \simeq S_i \oplus IH_p^{i+s+1}(X) \quad \text{for } i < 0,\end{aligned}$$

where $IH_p^{i+s+1}(X)$ is the primitive part of $IH^{i+s+1}(X)$ and where the last isomorphism follows from [KW06, Chapter 4.10]. Therefore we have

$$\begin{aligned} S_i &\simeq IH^{i+s+1}(X) \quad \text{for } i \geq 0, \\ S_i &\simeq L(IH^{i+s-1}(X)) \simeq IH^{i+s-1}(X) \quad \text{for } i < 0, \end{aligned}$$

where $L : IH^{i+s-1}(X) \rightarrow IH^{i+s+1}(X)$ is the Lefschetz operator which is injective for $i \leq 0$. \square

In the next proposition we show that at a generic point $\underline{\lambda} \in V$ the Radon transformation $\mathcal{R}(\mathcal{M}^{IC}(X))$ of $\mathcal{M}^{IC}(X)$ measures the intersection cohomology of $X \cap H_{\underline{\lambda}}$, where $H_{\underline{\lambda}}$ is the hyperplane in $\mathbb{P}(V')$ corresponding to $\underline{\lambda}$.

Proposition 2.14. *Let $\underline{\lambda}$ be a generic point of V and denote by $i_{\underline{\lambda}} : \{\underline{\lambda}\} \rightarrow V$ its embedding. We have the following isomorphism*

$$i_{\underline{\lambda}}^+ \mathcal{R}(\mathcal{M}^{IC}(X)) \simeq R\Gamma(X \cap H_{\underline{\lambda}}, IC_{X \cap H_{\underline{\lambda}}}),$$

in particular

$$\mathcal{H}^j(i_{\underline{\lambda}}^+ \mathcal{R}(\mathcal{M}^{IC}(X))) \simeq IH^{j+s-1}(X \cap H_{\underline{\lambda}}).$$

Proof. Consider the following diagram where all squares are cartesian

$$\begin{array}{ccccc} X & \xleftarrow{\pi_1^X} & Z_X & \xleftarrow{i_X} & X \cap H_{\underline{\lambda}} \\ \downarrow i & & \downarrow \eta & & \downarrow \eta^H \\ \mathbb{P}(V') & \xleftarrow{\pi_1^Z} & Z & \xleftarrow{i_H} & H_{\underline{\lambda}} \\ & & \downarrow \pi_2^Z & & \downarrow \pi^H \\ & & V & \xleftarrow{i_{\underline{\lambda}}} & \{\underline{\lambda}\} \end{array}$$

We have

$$\begin{aligned} DR(i_{\underline{\lambda}}^+ \mathcal{R}(\mathcal{M}^{IC}(X))) &\simeq i_{\underline{\lambda}}^! R\pi_{2*}(\pi_1^Z)^! i_! IC(X)[1] \\ &\simeq i_{\underline{\lambda}}^! R\pi_{2*} R\eta_* \pi_1^{X!} IC(X)[1] \\ &\simeq R\pi_*^H i_H^! R\eta_* \pi_1^{X!} IC(X)[1] \\ &\simeq R\pi_*^H R\eta_*^H i_X^! \pi_1^{X!} IC(X)[1] \\ &\simeq R(\pi^H \circ \eta^H)_*(\pi_1^X \circ i_X)^! IC(X)[1] \\ &\simeq R(\pi^H \circ \eta^H)_* IC(X \cap H_{\underline{\lambda}}) \\ &\simeq R\Gamma(X \cap H_{\underline{\lambda}}, IC(X \cap H_{\underline{\lambda}})), \end{aligned}$$

where the first isomorphism follows from $DR \circ i_{\underline{\lambda}}^+ = i_{\underline{\lambda}}^! \circ DR[t+1]$ and $DR \circ (\pi_1^Z)^+ \simeq (\pi_1^Z)^! \circ DR[-t]$ (see e.g. [HTT08, Theorem 7.1.1]), the second, third and fourth isomorphism follows from base change (see e.g. [Dim04, Theorem 3.2.13(ii)] and the sixth isomorphism follows from [GM83, Section 5.4.1] (notice that their $IC(X)$ is our $IC(X)[n]$ where $n = \dim_{\mathbb{C}}(X)$) and the fact that for a generic $\underline{\lambda}$ the hyperplane $H_{\underline{\lambda}}$ is transversal to a given Whitney stratification of X . The first claim now follows from the fact that the de Rham functor DR is the identity on a point. The second claim follows from $\mathbb{H}^{j-s+1}(X \cap H_{\underline{\lambda}}, IC(X \cap H_{\underline{\lambda}})) \simeq IH^j(X \cap H_{\underline{\lambda}})$. \square

Remark 2.15. *Combining Proposition 2.13 and Proposition 2.14 we see that we have the following decomposition for generic $\underline{\lambda} \in V$:*

$$IH^{s-1}(X \cap H_{\underline{\lambda}}) \simeq \mathcal{H}^0(i_{\underline{\lambda}}^+ \mathcal{R}(\mathcal{M}^{IC}(X))) \simeq i_{\underline{\lambda}}^+ \mathcal{H}^0(\mathcal{R}(\mathcal{M}^{IC}(X))) \simeq i_{\underline{\lambda}}^+ \mathcal{M}^{IC}(X^\circ, \mathcal{L}) \oplus IH^{s-1}(X).$$

This is the intersection cohomology analogon of the decomposition of the cohomology of a smooth hyperplane section of a smooth projective variety into its vanishing part and the ambient part.

We will now show that $\mathcal{M}^{IC}(X^\circ, \mathcal{L})$ can be expressed as an image of a morphism between GKZ-systems.

Theorem 2.16. *Let $\tilde{\beta}, \tilde{\beta}'$ as in Theorem 2.11, then $\mathcal{M}^{IC}(X^\circ, \mathcal{L}) \simeq \text{im}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'} \xrightarrow{\cdot \partial^{\tilde{\beta} + \tilde{\beta}'}} \mathcal{M}_{\tilde{B}}^{\tilde{\beta}})$.*

Proof. First recall that we have shown in the proof of Proposition 2.13. that $\mathcal{M}^{IC}(X^\circ, \mathcal{L}) \simeq \text{FL}(\mathcal{M}^{IC}(Y))$. On the other hand, as Y is the closure in V' of the image of the morphism h , the module $\mathcal{M}^{IC}(Y)$ is isomorphic to the image of $h_+ \mathcal{O}_T \rightarrow h_+ \mathcal{O}_T$. As the Fourier-Laplace transformation is exact we can conclude that $\mathcal{M}^{IC}(X^\circ, \mathcal{L})$ is isomorphic to the image of $\text{FL}(h_+ \mathcal{O}_T) \rightarrow \text{FL}(h_+ \mathcal{O}_T)$.

By Lemma 2.10 we know that $\text{FL}(h_+ \mathcal{O}_T)$ is isomorphic to $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ for every $\tilde{\beta} \in \delta_{\tilde{B}} + (\mathbb{R}_+ \tilde{B} \cap \mathbb{Z}^{s+1})$ and that $\text{FL}(h_+ \mathcal{O}_T)$ is isomorphic to $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}$ for every $\tilde{\beta}' \in (\mathbb{R}_+ \tilde{B})^\circ \cap \mathbb{Z}^{s+1}$. It follows now from the last statement of Lemma 2.10, that the induced morphism between $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}$ and $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ is equal to $\cdot \partial^{\tilde{\beta} + \tilde{\beta}'}$ up to some non-zero constant. \square

Denote by \mathcal{K} the kernel of the morphism $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'} \xrightarrow{\cdot \partial^{\tilde{\beta} + \tilde{\beta}'}} \mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$, then $\mathcal{M}^{IC}(X^\circ, \mathcal{L})$ is isomorphic to the quotient $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}/\mathcal{K}$ in the abelian category of regular holonomic \mathcal{D} -modules. The next result gives a concrete description of \mathcal{K} as a submodule of $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}$.

First, we define a sub- D_V -module $\Gamma_{\partial, c}(M_{\tilde{B}}^{-\tilde{\beta}'})$ of $M_{\tilde{B}}^{-\tilde{\beta}'}$, where $c \in \rho^{-1}(\tilde{\beta} + \tilde{\beta}')$ (cf. Equation (7)) :

$$\Gamma_{\partial, c}(M_{\tilde{B}}^{-\tilde{\beta}'}) := \{m \in M_{\tilde{B}}^{-\tilde{\beta}'} \mid \exists n \in \mathbb{N} \text{ with } (\partial^c)^n \cdot m = 0\}$$

Recall that two elements ∂^{c_1} and ∂^{c_2} with $c_1, c_2 \in \rho^{-1}(\tilde{\beta} + \tilde{\beta}')$ differ by some element $P \cdot \square_{\underline{l}}$, where $P \in \mathbb{C}[\partial_0, \dots, \partial_s]$ and $\underline{l} = c_1 - c_2$. Any element $m \in M_{\tilde{B}}^{-\tilde{\beta}'}$ is eliminated by left multiplication with some high enough power of $P \cdot \square_{\underline{l}}$. This shows that $\Gamma_{\partial, c}(M_{\tilde{B}}^{-\tilde{\beta}'})$ is actually independent of the chosen $c \in \rho^{-1}(\tilde{\beta} + \tilde{\beta}')$. Thus we denote it just by $\Gamma_{\partial}(M_{\tilde{B}}^{-\tilde{\beta}'})$ and the corresponding sub- \mathcal{D}_V -module of $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}$ by $\Gamma_{\partial}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'})$.

Proposition 2.17. *Let $\tilde{\beta}, \tilde{\beta}'$ as in Theorem 2.11 and let \mathcal{K} be the kernel of $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'} \xrightarrow{\cdot \partial^{\tilde{\beta} + \tilde{\beta}'}} \mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$. Then*

$$\mathcal{K} \simeq \Gamma_{\partial}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}), \quad \text{in particular} \quad \mathcal{M}^{IC}(X^\circ, \mathcal{L}) \simeq \mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}/\Gamma_{\partial}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}).$$

Proof. Recall that the morphism $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'} \xrightarrow{\cdot \partial^{\tilde{\beta} + \tilde{\beta}'}} \mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ is induced by the morphism $\text{FL}(h_+ \mathcal{O}_T) \rightarrow \text{FL}(h_+ \mathcal{O}_T)$, where we used the isomorphisms $\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'} \simeq \text{FL}(h_+ \mathcal{O}_T)$ and $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}} \simeq \text{FL}(h_+ \mathcal{O}_T)$. Applying the Fourier-Laplace transformation again and using $\text{FL} \circ \text{FL} = \text{Id}$, we see that the morphism $\text{FL}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}) \xrightarrow{\cdot w^{\tilde{\beta} + \tilde{\beta}'}} \text{FL}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}})$ is induced by the morphism $h_+ \mathcal{O}_T \rightarrow h_+ \mathcal{O}_T$. We will calculate the kernel of $\text{FL}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}) \xrightarrow{\cdot w^{\tilde{\beta} + \tilde{\beta}'}} \text{FL}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}})$. First notice that the map h can be factorized as $h = k \circ l$, where k is the canonical inclusion of $(\mathbb{C}^*)^{t+1} \rightarrow V'$ and the map l is given by

$$l : T \longrightarrow (\mathbb{C}^*)^{t+1}, \\ (y_0, \dots, y_r) \mapsto (\underline{y}^{\tilde{b}_0}, \dots, \underline{y}^{\tilde{b}_t}) = (y_0, y_0 \underline{y}^{\tilde{b}_1}, \dots, y_0 \underline{y}^{\tilde{b}_t}).$$

This shows that $\text{FL}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}) \simeq k_+ l_+ \mathcal{O}_T$ is localized along $V' \setminus (\mathbb{C}^*)^{t+1}$, i.e. $\text{FL}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}) \simeq k_+ k^+ \text{FL}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}})$. Let $D_1 = \{w^{\tilde{\beta} + \tilde{\beta}'} = 0\}$, set $U_1 := V' \setminus D_1$ and denote by $j_1 : U_1 \rightarrow V'$ the canonical inclusion. Because $(\mathbb{C}^*)^{t+1} \subset U_1$, the \mathcal{D} -module $\text{FL}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}})$ is also localized along D_1 , i.e. $\text{FL}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}) \simeq j_{1+} j_1^+ \text{FL}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}})$. Notice that the induced morphism $j_1^+ \text{FL}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}) \rightarrow j_1^+ \text{FL}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}})$ is an isomorphism, because $w^{\tilde{\beta} + \tilde{\beta}'}$ is invertible on U_1 . Therefore we can conclude that $j_{1+} j_1^+ \text{FL}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}) \rightarrow j_{1+} j_1^+ \text{FL}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}) \simeq \text{FL}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}})$ is an

isomorphism. It is therefore enough to calculate the kernel of $\mathrm{FL}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}) \rightarrow j_{1+}j_1^+ \mathrm{FL}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'})$. On the level of global sections this is $H_{D_1}^0(\mathrm{FL}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}))$ (cf. [HTT08, Proposition 1.7.1]) which is given by

$$H_{D_1}^0(\mathrm{FL}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'})) = \{m \in \mathrm{FL}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'}) \mid \exists n \in \mathbb{N} \text{ with } (w^{\tilde{\beta}+\tilde{\beta}'})^n \cdot m = 0\}.$$

Applying the Fourier-Laplace transformation to this kernel shows the claim. \square

2.4 The equivariant setting

In this section we show that the \mathcal{D} -modules discussed above are quasi-equivariant with respect to a natural torus action. We review the definition of an quasi-equivariant \mathcal{D} -modules from [Kas08, Chapter 3] and prove some simple statements for these.

Let \mathcal{X} be smooth, complex, quasi-projective variety and G be a complex affine algebraic group, which acts on \mathcal{X} . Denote by $\nu : G \times \mathcal{X} \rightarrow \mathcal{X}$ the action of G on \mathcal{X} and by $p_2 : G \times \mathcal{X} \rightarrow \mathcal{X}$ the second projection. A $\mathcal{D}_{\mathcal{X}}$ -module \mathcal{M} is called quasi- G -equivariant if it satisfies $\nu^+ \mathcal{M} \simeq p_2^+ \mathcal{M}$ as $\mathcal{O}_G \boxtimes \mathcal{D}_{\mathcal{X}}$ -modules together with a associate law (cf. [Kas08, Definition 3.1.3]). We denote the abelian category of quasi- G -equivariant $\mathcal{D}_{\mathcal{X}}$ -modules by $M(\mathcal{D}_{\mathcal{X}}, G)$ and the subcategories of coherent, holonomic and regular holonomic quasi- G -equivariant $\mathcal{D}_{\mathcal{Y}}$ -modules by $M_{coh}(\mathcal{D}_{\mathcal{X}}, G)$ resp. $M_h(\mathcal{D}_{\mathcal{X}}, G)$ resp. $M_{rh}(\mathcal{D}_{\mathcal{X}}, G)$. The corresponding bounded derived categories are denoted by $D_*^b(\mathcal{D}_{\mathcal{X}}, G)$ for $*$ = \emptyset, coh, h, rh .

A $\mathcal{O}_{\mathcal{X}}$ -module \mathcal{F} is called G -equivariant if $\nu^* \mathcal{F} \simeq pr^* \mathcal{F}$ as $\mathcal{O}_{G \times \mathcal{X}}$ -modules and if it satisfies an associative law (cf. [Kas08, Definition 3.1.2]). We denote by $Mod(\mathcal{O}_{\mathcal{X}}, G)$ the category of G -equivariant $\mathcal{O}_{\mathcal{X}}$ -modules and by $Mod_{coh}(\mathcal{O}_{\mathcal{X}}, G)$ the subcategory of coherent G -equivariant $\mathcal{O}_{\mathcal{X}}$ -modules.

Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a G -equivariant map. Then the direct image resp. the inverse image functors preserve quasi- G -equivariance (cf. [Kas08, Equation (3.4.1), Equation (3.5.2)]).

We will now show that the duality functor preserves quasi- G -equivariance.

Proposition 2.18. *Let $M \in D_{coh}^b(\mathcal{D}_{\mathcal{X}}, G)$ then $\mathbb{D}(M) \in D_{coh}^b(\mathcal{D}_{\mathcal{X}}, G)^{opp}$.*

Proof. By a dévissage we may assume that M is a single degree complex, i.e. $M \in Mod_{coh}(\mathcal{D}_{\mathcal{X}}, G)$. By [Kas08, Lemma 3.3.2] for every $N \in Mod_{coh}(\mathcal{O}_{\mathcal{X}}, G)$ there exists a G -equivariant locally-free $\mathcal{O}_{\mathcal{X}}$ module L of finite rank and a surjective G -equivariant morphism $L \twoheadrightarrow N$. Notice that there exists a G -equivariant coherent $\mathcal{O}_{\mathcal{X}}$ -submodule K of M with $\mathcal{D}_{\mathcal{X}} \otimes K = M$. This enables us to construct a locally-free, G -equivariant resolution

$$\cdots \rightarrow L_2 \rightarrow L_1 \rightarrow L_0 \rightarrow K \rightarrow 0$$

of K in $Mod_{coh}(\mathcal{O}_{\mathcal{X}}, G)$, which gives rise to a resolution of M

$$\cdots \rightarrow \mathcal{D}_{\mathcal{X}} \otimes L_2 \rightarrow \mathcal{D}_{\mathcal{X}} \otimes L_1 \rightarrow \mathcal{D}_{\mathcal{X}} \otimes L_0 \rightarrow M \rightarrow 0$$

in $Mod_{coh}(\mathcal{D}_{\mathcal{X}}, G)$ by the exactness of $\mathcal{D}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}}$. We have

$$\begin{aligned} \mathbb{D}M &= R\mathrm{Hom}_{\mathcal{D}_{\mathcal{X}}}(M, \mathcal{D}_{\mathcal{X}}) \otimes \Omega_{\mathcal{X}}^{\otimes -1}[dim \mathcal{X}] \\ &\simeq \mathrm{Hom}_{\mathcal{D}_{\mathcal{X}}}(\mathcal{D}_{\mathcal{X}} \otimes L_{\bullet}, \mathcal{D}_{\mathcal{X}}) \otimes \Omega_{\mathcal{X}}^{\otimes -1}[dim \mathcal{X}] \\ &\simeq (\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(L_{\bullet}, \mathcal{O}_{\mathcal{X}}) \otimes \mathcal{D}_{\mathcal{X}}) \otimes \Omega_{\mathcal{X}}^{\otimes -1}[dim \mathcal{X}] \\ &\simeq \mathcal{D}_{\mathcal{X}} \otimes \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(L_{\bullet}, \mathcal{O}_{\mathcal{X}})[dim \mathcal{X}]. \end{aligned}$$

But $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(L_{\bullet}, \mathcal{O}_{\mathcal{X}})$ is again a complex in $Mod_{coh}(\mathcal{O}_{\mathcal{X}}, G)$, which can be easily seen by the local-freeness of the L_i . Thus we can conclude that $\mathbb{D}M \in D_{coh}^b(\mathcal{D}_{\mathcal{X}}, G)^{opp}$. \square

Corollary 2.19. *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a G -equivariant map. Then the proper direct image and the exceptional inverse image functor preserves quasi- G -equivariance.*

Proof. This follows from $f_{\dagger} = \mathbb{D} \circ f_+ \circ \mathbb{D}$ and $f^{\dagger} = \mathbb{D} \circ f^+ \circ \mathbb{D}$. \square

In the next proposition we will show that the characteristic variety of a quasi- G -equivariant \mathcal{D} -module is G -invariant. For that purpose, we will consider the action induced by ν on the cotangent bundle $T^*\mathcal{X}$. More precisely, consider the differential $d\nu$ of the action map, which is a map of vector bundles $d\nu : \nu^*T^*\mathcal{X} \rightarrow T^*(G \times \mathcal{X}) = T^*G \boxtimes T^*\mathcal{X}$ over $G \times \mathcal{X}$, or, equivalently, a map $d\nu : (G \times \mathcal{X}) \times_{\mathcal{X}} T^*\mathcal{X} \rightarrow T^*G \times T^*\mathcal{X}$ of smooth complex varieties. Notice that

$$\begin{aligned} t : G \times T^*\mathcal{X} &\longrightarrow (G \times \mathcal{X}) \times_{\mathcal{X}} T^*\mathcal{X} = \{((g, x), v) \mid \pi(v) = \nu(g, x)\mathcal{X}\}, \\ (g, v) &\longmapsto (g, \nu(g^{-1}, \pi(v)), v) \end{aligned}$$

is an isomorphism, with inverse map sending $((g, x), v)$ to (g, v) . Now consider the composition $\xi : \tilde{p}_2 \circ d\nu \circ t : G \times T^*\mathcal{X} \rightarrow T^*\mathcal{X}$, where $\tilde{p}_2 : T^*G \times T^*\mathcal{X} \rightarrow T^*\mathcal{X}$ is the second projection. One easily checks that we have $\xi(g_1 \cdot g_2, x) = \xi(g_1, \xi(g_2, x))$, i.e., that we obtain an action of G on $T^*\mathcal{X}$. Notice that for any $g \in G$, the map $\xi(g, -) : T^*\mathcal{X} \rightarrow T^*\mathcal{X}$ is nothing but the differential $d\nu_g$ of the map $\nu_g : \mathcal{X} \rightarrow \mathcal{X}$ where $\nu_g(x) := \nu(g, x)$. Notice that for $M \in D^b(\mathcal{D}_{\mathcal{X}}, G)$ one has $\nu_g^+M \simeq M$ by the quasi- G -equivariance of M .

Proposition 2.20. *Let $M \in D_{coh}^b(\mathcal{D}_{\mathcal{X}}, G)$, then the characteristic variety $\text{char}(M)$ of M is invariant under the G -action on $T^*\mathcal{X}$ given by ξ . Moreover, if G is irreducible then the irreducible components of $\text{char}(M)$ are also G -invariant.*

Proof. For both statements it is sufficient to show invariance under the morphism ν_g for any $g \in G$. We are going to use the following fact (cf. [HTT08, Lemma 2.4.6(iii)]). Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism between smooth algebraic varieties. One has the natural morphisms

$$T^*\mathcal{X} \xleftarrow{\rho_f} \mathcal{X} \times_{\mathcal{Y}} T^*\mathcal{Y} \xrightarrow{\omega_f} T^*\mathcal{Y}.$$

Let $M \in \text{Mod}_{coh}(\mathcal{D}_{\mathcal{Y}})$. If f is non-characteristic then $\text{char}(f^+M) \subset \rho_f \omega_f^{-1}(\text{char}(M))$.

We want to apply this to the case $f = \nu_g$. Notice that in this case the maps ρ_{ν_g} and ω_{ν_g} are isomorphisms and $\rho_{\nu_g} \circ \omega_{\nu_g}^{-1} = d\nu_g$. Thus we have

$$\text{char}(M) = \text{char}(\nu_g^+M) \subset d\nu_g(\text{char}(M)).$$

Repeating the argument with $\nu_{g^{-1}}$ gives $\text{char}(M) \subset d\nu_{g^{-1}}(\text{char}(M))$. Now applying $d\nu_g$ to both sides of the latter inclusion shows the first claim.

Now assume that G is irreducible and let C_i be an irreducible component of $\text{Ch}(M)$. Notice that $G \times C_i$ is irreducible. Consider the scheme-theoretic image I of $G \times C_i$ under the induced action map $\xi : G \times \text{char}(M) \rightarrow \text{char}(M)$. Then $\bar{\xi} : G \times C_i \rightarrow I$ is a dominant morphism. We want to show that I is irreducible. Let $U \subset I$ be an affine open set. The restriction $\bar{\xi}^{-1}(U) \rightarrow U$ is still dominant and induces an injective ring homomorphism $\mathcal{O}_I(U) \rightarrow \mathcal{O}_{G \times C_i}(\bar{\xi}^{-1}(U))$. As $G \times C_i$ is irreducible and reduced the ring $\mathcal{O}_{G \times C_i}(\bar{\xi}^{-1}(U))$ is a domain. Thus $\mathcal{O}_I(U)$ is also a domain and because U was chosen arbitrary we conclude that I is irreducible. Notice that we have $C_i \subset I \subset \text{char}(M)$ and therefore $C_i = I$, which shows the claim. \square

The proposition above enables us to prove that a section of a quotient map of a free action is non-characteristic with respect to quasi- G -equivariant \mathcal{D} -modules.

Lemma 2.21. *Let $G \times \mathcal{X} \rightarrow \mathcal{X}$ be a free action and $\pi_G : \mathcal{X} \rightarrow \mathcal{X}/G$ a geometric quotient. Let $i_G : \mathcal{X}/G \rightarrow \mathcal{X}$ be a section of π_G , then i_G is non-characteristic with respect to every $M \in D_{rh}^b(G, \mathcal{D}_{\mathcal{X}})$.*

Proof. We consider \mathcal{X}/G as smooth subvariety of \mathcal{X} . Notice that \mathcal{X}/G is transversal to the orbits of the G -action on \mathcal{X} given by ν . Let $\text{char}(M) = \bigcup_{i \in I} C_i$ be the decomposition into irreducible components and put $\mathcal{X}_i := \pi(\mathcal{X}_i)$ so that $C_i = T_{\mathcal{X}_i}^*\mathcal{X}$. From Proposition 2.20 we know that C_i is invariant under the action given by ξ , and hence a union of orbits of this G -action. On the other hand, the image under the projection $\pi : T^*\mathcal{X} \rightarrow \mathcal{X}$ of such an orbit is necessarily an orbit of the original action given by

ν . Hence \mathcal{X}_i is a union $\bigcup_j \mathcal{X}_i^{(j)}$ of G -orbits, more precisely, these orbits form a Whitney stratification of \mathcal{X}_i (see, [Dim92, Proposition 1.14]). Whitney's condition A then implies that $T_{\mathcal{X}_i}^* \mathcal{X} \subset \bigcup_j T_{\mathcal{X}_i^{(j)}}^* \mathcal{X}_i$.

Transversality of \mathcal{X}/G and the orbits $\mathcal{X}_i^{(j)}$ means that $T_{\mathcal{X}/G}^* \mathcal{X} \cap T_{\mathcal{X}_i^{(j)}}^* \mathcal{X}_i \subset T_{\mathcal{X}}^* \mathcal{X}$, from which we deduce that $T_{\mathcal{X}/G}^* \mathcal{X} \cap T_{\mathcal{X}_i}^* \mathcal{X}_i \subset T_{\mathcal{X}}^* \mathcal{X}$ and hence $T_{\mathcal{X}/G}^* \mathcal{X} \cap \text{char}(M) \subset T_{\mathcal{X}}^* \mathcal{X}$. Thus i_G non-characteristic with respect to M as required. \square

Let $V^* = \mathbb{C} \times (\mathbb{C}^*)^t$ and let $j_{V^*} : V^* \rightarrow V$ be the canonical embedding. Consider the following diagram

$$\begin{array}{ccccc}
S & \xleftarrow{\pi_1^S} & \Gamma & \xleftarrow{j_{\Gamma^*}} & \Gamma^* \\
j \downarrow & & \theta \downarrow & & \zeta \downarrow \\
X & \xleftarrow{\quad} & Z_X & \xleftarrow{j_{Z_X^*}} & Z_X^* \\
i \downarrow & & \eta \downarrow & & \varepsilon \downarrow \\
\mathbb{P}(V') & \xleftarrow{\pi_1^Z} & Z & \xleftarrow{j_{Z^*}} & Z^* \\
& & \pi_2^Z \downarrow & & \delta \downarrow \\
& & V & \xleftarrow{j_{V^*}} & V^*
\end{array} \tag{19}$$

where the varieties Z^*, Z_X^*, Γ^* together with the maps $j_{Z^*}, j_{Z_X^*}, j_{\Gamma^*}$ and $\delta, \varepsilon, \zeta$ are induced by the base change j_{V^*} . Thus all squares in the diagram above are cartesian.

We now specify to the case $G = (\mathbb{C}^*)^s$. We let G act on S and V by

$$\begin{aligned}
G \times S &\longrightarrow S, \\
(g_1, \dots, g_s, y_1, \dots, y_s) &\mapsto (g_1 y_1, \dots, g_s y_s), \\
G \times V &\longrightarrow V, \\
(g_1, \dots, g_s, \lambda_0, \dots, \lambda_t) &\mapsto (\lambda_0, \underline{g}^{-b_1} \lambda_1, \dots, \underline{g}^{-b_t} \lambda_t).
\end{aligned} \tag{20}$$

We also define the following G -action on $\mathbb{P}(V')$:

$$\begin{aligned}
G \times \mathbb{P}(V') &\longrightarrow \mathbb{P}(V'), \\
(g_1, \dots, g_s, (\mu_0 : \dots : \mu_t)) &\mapsto (\mu_0 : \underline{g}^{b_1} \mu_1 : \dots : \underline{g}^{b_t} \mu_t).
\end{aligned} \tag{21}$$

This makes map $g = i \circ j : S \rightarrow \mathbb{P}(V')$ G -equivariant. There is a natural action of G on $\mathbb{P}(V') \times V$ resp. $S \times V$ which leaves the subvarieties $Z = \{\sum_{i=0}^t \lambda_i \mu_i = 0\}$ resp. $\Gamma = \{\lambda_0 + \sum_{i=1}^t \lambda_i \underline{y}^{b_i}\}$ invariant. It is now easy to see, using the induced actions on Γ resp. Z , that the maps $\pi_1^Z, \pi_2^Z, \pi_1^S$ as well as η and θ are G -equivariant.

Notice that G leaves V^* invariant and acts freely on it, but this shows that G acts also freely on Z^*, Z_X^* and Γ^* . Therefore also the maps $\delta, \varepsilon, \zeta$ are G -equivariant. Notice that the action of G on $\mathbb{P}(V')$ as defined in formula (21) is not free, there are orbits of dimension strictly smaller dimension than $s = \dim(G)$. Because we have $\mathbb{Z}B = \mathbb{Z}^s$, there exist matrices $N_1 \in Gl(s \times s, \mathbb{Z})$ and $N_2 \in Gl(t \times t, \mathbb{Z})$ such that

$$B = N_1 \cdot (I_s \mid 0_{s \times r}) \cdot N_2,$$

where $r := t - s$. Define matrices

$$L := N_2^{-1} \cdot \begin{pmatrix} 0_{s \times r} \\ I_r \end{pmatrix}, \quad M := (0_{r \times s} \mid I_r) \cdot N_2, \quad C := N_2^{-1} \cdot \begin{pmatrix} I_s \\ 0_{r \times s} \end{pmatrix} \cdot N_1^{-1}, \quad D := (C \cdot B)^t,$$

whose entries we denote by l_{ij}, m_{ji}, c_{ik} and d_{il} , respectively. Then $M \cdot L = I_r, B \cdot C = I_s, B \cdot L = 0, M \cdot C = 0$ and

$$C \cdot B + L \cdot M = I_t. \tag{22}$$

Consider the following map, where $F := (\mathbb{C}^*)^s$:

$$T_{\mathbb{P}} : \mathbb{P}(V') \times \mathbb{C} \times F \times \mathcal{KM} \longrightarrow \mathbb{P}(V') \times V^*,$$

$$((\mu_0 : \dots : \mu_t), \lambda_0, f_1, \dots, f_s, q_1, \dots, q_r) \mapsto ((\mu_0 : \underline{f}^{-b_1} \mu_1 : \dots : \underline{f}^{-b_t} \mu_t), \lambda_0, \underline{f}^{b_1} \cdot \underline{q}^{m_1}, \dots, \underline{f}^{b_t} \cdot \underline{q}^{m_t})$$

with $\underline{f}^{b_i} = \prod_{k=1}^s f_k^{b_{ki}}$, $\underline{q}^{m_i} = \prod_{j=1}^r q_j^{m_{ji}}$ and inverse

$$T_{\mathbb{P}}^{-1} : \mathbb{P}(V') \times V^* \longrightarrow \mathbb{P}(V') \times \mathbb{C} \times F \times \mathcal{KM},$$

$$((\mu_0 : \dots : \mu_t), \lambda_0, \dots, \lambda_t) \mapsto ((\mu_0 : \lambda^{d_1} \mu_1 : \dots : \lambda^{d_t} \mu_t), \lambda_0, \underline{\lambda}^{c_1}, \dots, \underline{\lambda}^{c_s}, \underline{\lambda}^{l_1}, \dots, \underline{\lambda}^{l_r})$$

with $\underline{\lambda}^{c_k} := \prod_{i=1}^t \lambda_i^{c_{ik}}$, $\underline{\lambda}^{l_j} = \prod_{i=1}^t \lambda_i^{l_{ij}}$ and $\lambda^{d_i} := \prod_{i=1}^t \lambda_i^{d_{ii}} = \prod_{i=1}^t \lambda_i^{\sum_k c_{ik} b_{ki}}$.

Recall the following G -action on $\mathbb{P}(V') \times V^*$

$$G \times (\mathbb{P}(V') \times V^*) \longrightarrow \mathbb{P}(V') \times V^*,$$

$$(g_1, \dots, g_s, (\mu_0 : \dots : \mu_t), \lambda_0, \dots, \lambda_t) \mapsto ((\mu_0 : \underline{g}^{b_1} \mu_1 : \dots : \underline{g}^{b_t} \mu_t), \lambda_0, \underline{g}^{-b_1} \lambda_1, \dots, \underline{g}^{-b_t} \lambda_t).$$

Consider the following G -action on $\mathbb{P}(V') \times \mathbb{C} \times F \times \mathcal{KM}$

$$G \times (\mathbb{P}(V') \times \mathbb{C} \times F \times \mathcal{KM}) \longrightarrow \mathbb{P}(V') \times \mathbb{C} \times F \times \mathcal{KM},$$

$$(g_1, \dots, g_s, (\mu_0 : \dots : \mu_t), \lambda_0, f_1, \dots, f_s, q_1, \dots, q_r) \mapsto ((\mu_0 : \mu_1 : \dots : \mu_t), \lambda_0, g_1^{-1} f_1, \dots, g_s^{-1} f_s, q_1, \dots, q_r).$$

It is easy to see that $T_{\mathbb{P}}$ resp. $T_{\mathbb{P}}^{-1}$ is G -equivariant with respect to the G -actions above.

Consider the map

$$T_S : S \times \mathbb{C} \times F \times \mathcal{KM} \longrightarrow S \times V^*,$$

$$(y_1, \dots, y_s, \lambda_0, f_1, \dots, f_s, q_1, \dots, q_r) \mapsto (f_1^{-1} y_1, \dots, f_s^{-1} y_s, \lambda_0, \underline{f}^{b_1} \cdot \underline{q}^{m_1}, \dots, \underline{f}^{b_t} \cdot \underline{q}^{m_t})$$

and its inverse

$$T_S^{-1} : S \times V^* \longrightarrow S \times \mathbb{C} \times F \times \mathcal{KM},$$

$$(y_1, \dots, y_s, \lambda_0, \dots, \lambda_t) \mapsto (\underline{\lambda}^{c_1} y_1, \dots, \underline{\lambda}^{c_s} y_s, \lambda_0, \underline{\lambda}^{c_1}, \dots, \underline{\lambda}^{c_s}, \underline{\lambda}^{l_1}, \dots, \underline{\lambda}^{l_r}),$$

where one has to use (22).

Recall the G -action on $S \times V^*$

$$G \times (S \times V^*) \longrightarrow S \times V^*,$$

$$(g_1, \dots, g_s, \lambda_0, \dots, \lambda_t) \mapsto (g_1 y_1, \dots, g_s y_s, \lambda_0, \underline{g}^{-b_1} \lambda_1, \dots, \underline{g}^{-b_t} \lambda_t)$$

and consider the following G -action on $S \times \mathbb{C} \times F \times \mathcal{KM}$

$$G \times (S \times \mathbb{C} \times F \times \mathcal{KM}) \longrightarrow S \times \mathbb{C} \times F \times \mathcal{KM},$$

$$(g_1, \dots, g_s, y_1, \dots, y_s, \lambda_0, f_1, \dots, f_s, q_1, \dots, q_r) \mapsto (y_1, \dots, y_s, \lambda_0, g_1^{-1} f_1, \dots, g_s^{-1} f_s, q_1, \dots, q_r).$$

It is again easy to see that T_S resp. T_S^{-1} is G -equivariant with respect to the G -actions above.

The subvarieties Z^* resp. Γ^* are then given by $\lambda_0 \mu_0 + \sum_{i=1}^t \mu_i \cdot \underline{q}^{m_i} = 0$ resp. $\lambda_0 + \sum_{i=1}^t \underline{y}^{b_i} \cdot \underline{q}^{m_i} = 0$.

Finally consider the maps

$$T : \mathbb{C} \times F \times \mathcal{KM} \longrightarrow V^*,$$

$$(\lambda_0, f_1, \dots, f_s, q_1, \dots, q_r) \mapsto (\lambda_0, \underline{f}^{b_1} \cdot \underline{q}^{m_1}, \dots, \underline{f}^{b_t} \cdot \underline{q}^{m_t}),$$

$$T^{-1} : V^* \longrightarrow \mathbb{C} \times F \times \mathcal{KM},$$

$$(\lambda_0, \dots, \lambda_t) \mapsto (\lambda_0, \underline{\lambda}^{c_1}, \dots, \underline{\lambda}^{c_s}, \underline{\lambda}^{l_1}, \dots, \underline{\lambda}^{l_r}),$$

which are G -equivariant with respect to the G -action on V^* and the following G -action on $\mathbb{C} \times F \times \mathcal{KM}$

$$G \times (\mathbb{C} \times F \times \mathcal{KM}) \longrightarrow \mathbb{C} \times F \times \mathcal{KM},$$

$$(g_1, \dots, g_s, \lambda_0, f_1, \dots, f_s, q_1, \dots, q_r) \mapsto (\lambda_0, g_1^{-1} f_1, \dots, g_s^{-1} f_s, q_1, \dots, q_r).$$

The G -equivariant isomorphisms above show that the geometric quotients of V^* , Z^* and Γ^* by G exist and are given by $\mathbb{C} \times \mathcal{KM}$,

$$\mathcal{Z} := \{\lambda_0 \mu_0 + \sum_{i=1}^t q_i^{m_i} \mu_i = 0\} \subset \mathbb{P}(V') \times \mathbb{C} \times \mathcal{KM}$$

and

$$\mathcal{G} := \{\lambda_0 + \sum_{i=1}^t q_i^{m_i} y_{b_i} = 0\} \subset S \times \mathbb{C} \times \mathcal{KM},$$

respectively. We denote the corresponding quotient maps by $\pi_G^{V^*}$, $\pi_G^{Z^*}$ and $\pi_G^{\Gamma^*}$.

Notice that we have a natural section $i_G^{V^*}$ to $\pi_G^{V^*}$, which is induced by the inclusion

$$\mathbb{C} \times \mathcal{KM} \longrightarrow \mathbb{C} \times F \times \mathcal{KM},$$

$$(\lambda_0, q_1, \dots, q_r) \mapsto (\lambda_0, 1, \dots, 1, q_1, \dots, q_r)$$

and the isomorphism above. This gives also rise to sections $i_G^{Z^*}$ and $i_G^{\Gamma^*}$ of $\pi_G^{Z^*}$ resp. $\pi_G^{\Gamma^*}$. Consider the following diagram

$$\begin{array}{ccccccc} S & \xleftarrow{\pi_1^S} & \Gamma & \xleftarrow{j_{\Gamma^*}} & \Gamma^* & \xleftarrow{i_G^{\Gamma^*}} & \mathcal{G} \\ \downarrow j & & \downarrow \theta & & \downarrow \zeta & \xrightarrow{\pi_G^{\Gamma^*}} & \downarrow \gamma \\ X & \xleftarrow{} & Z_X & \xleftarrow{j_{Z_X^*}} & Z_X^* & \xleftarrow{i_G^{Z_X^*}} & Z_X \\ \downarrow i & & \downarrow \eta & & \downarrow \varepsilon & & \downarrow \beta \\ \mathbb{P}(V') & \xleftarrow{\pi_1^Z} & Z & \xleftarrow{j_{Z^*}} & Z^* & \xleftarrow{i_G^{Z^*}} & Z \\ & & \downarrow \pi_Z^2 & & \downarrow \delta & \xrightarrow{\pi_G^{Z^*}} & \downarrow \alpha \\ & & V & \xleftarrow{j_{V^*}} & V^* & \xleftarrow{i_G^{V^*}} & \mathbb{C} \times \mathcal{KM} \\ & & & & & \xrightarrow{\pi_G^{V^*}} & \end{array} \quad (23)$$

Notice also that all squares are cartesian.

Proposition 2.22. *Let $i_G^{Z^*} : Z \rightarrow Z^*$ resp. $i_G^{V^*} : \mathbb{C} \times \mathcal{KM} \rightarrow V^*$ the sections constructed above.*

1. The \mathcal{D}_{Z^*} -modules

$$(\varepsilon \circ \zeta)_+ \mathcal{O}_{\Gamma^*}, \quad (\varepsilon \circ \zeta)_+ \mathcal{O}_{\Gamma^*} \quad \text{and} \quad \mathcal{M}^{IC}(Z_X^*)$$

are quasi- G -equivariant and non-characteristic with respect to $i_G^{Z^*}$.

2. The \mathcal{D}_{V^*} -modules

$$\mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W^*}) \quad \text{and} \quad \mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W^*})$$

are quasi- G -equivariant and non-characteristic with respect to $i_G^{V^*}$.

3. We have

$$(i_G^{Z^*})_+ \mathcal{M}^{IC}(Z_X^*) \simeq \mathcal{M}^{IC}(Z_X).$$

In particular we have

$$\alpha_+ \mathcal{M}^{IC}(Z_X) \simeq i_{\mathcal{KM}}^+ \mathcal{R}(\mathcal{M}^{IC}(X)), \quad (24)$$

where $i_{\mathcal{KM}} := j_{V^*} \circ i_G^{V^*}$ is non-characteristic with respect to $\mathcal{R}(\mathcal{M}^{IC}(X))$.

Proof. 1. First notice that because the map $(i \circ j) : S \rightarrow \mathbb{P}(V')$ is affine and this property is preserved by base change, the map $(\varepsilon \circ \zeta)$ is also affine. Thus the direct image as well as the proper direct image of \mathcal{O}_{Γ^*} is a single \mathcal{D}_{Z^*} -module. The closure of Γ^* in Z^* is Z_X^* , therefore we have

$$\mathcal{M}^{IC}(Z_X^*) = \text{im}((\varepsilon \circ \zeta)_\dagger \mathcal{O}_{\Gamma^*} \rightarrow (\varepsilon \circ \zeta)_+ \mathcal{O}_{\Gamma^*}) \in \text{Mod}_{rh}(\mathcal{D}_{Z^*}). \quad (25)$$

To show the first claim, it is enough by Lemma 2.21 to show that the corresponding \mathcal{D} -modules are quasi- G -equivariant. First recall that $\Gamma^* \subset S \times V^*$ and denote by $\iota : \Gamma^* \rightarrow S$ the restriction of the projection to the first factor. Notice that ι is G -equivariant and $\mathcal{O}_{\Gamma^*} \simeq \iota^+ \mathcal{O}_S$. Therefore \mathcal{O}_{Γ^*} is a quasi- G -equivariant \mathcal{D} -module. Because ε, ζ is G -equivariant we see that $(\varepsilon \circ \zeta)_\dagger \mathcal{O}_{\Gamma^*}$ and $(\varepsilon \circ \zeta)_+ \mathcal{O}_{\Gamma^*}$ are quasi- G -equivariant. Furthermore, because of Equation (25) and the fact that $\text{Mod}(G, \mathcal{D}_{Z^*})$ is an abelian category the \mathcal{D} -module $\mathcal{M}^{IC}(Z_X^*)$ is quasi- G -equivariant.

2. For the second point, consider the action of G on $W^* = (\mathbb{C}^*)^t$ which is given by

$$\begin{aligned} G \times W^* &\longrightarrow W^*, \\ (g_1, \dots, g_s, \lambda_1, \dots, \lambda_t) &\mapsto (\underline{g}^{-b_1} \lambda_1, \dots, \underline{g}^{-b_t} \lambda_t). \end{aligned}$$

This action together with the action (20) induces a G -action on $S \times W^*$. It is easy to see that $\varphi_B|_{S \times W^*}$ is G -equivariant. Thus the \mathcal{D}_{V^*} -modules $\mathcal{H}^0(\varphi_{B,\dagger} \mathcal{O}_{S \times W^*})$ and $\mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W^*})$ are quasi- G -equivariant. The fact that $i_G^{V^*}$ is non-characteristic with respect to these \mathcal{D}_{V^*} -modules follows now again from Lemma 2.21.

3. To show the third claim, consider the following isomorphisms

$$\begin{aligned} \mathcal{M}^{IC}(\mathcal{Z}_X) &\simeq \text{im}((\beta \circ \gamma)_\dagger \mathcal{O}_G \rightarrow (\beta \circ \gamma)_+ \mathcal{O}_G) \\ &\simeq \text{im}\left((\beta \circ \gamma)_\dagger (i_G^{\Gamma^*})^\dagger \mathcal{O}_{\Gamma^*} \rightarrow (\beta \circ \gamma)_+ (i_G^{\Gamma^*})^+ \mathcal{O}_{\Gamma^*}\right) \\ &\simeq \text{im}\left((i_G^{Z^*})^\dagger (\varepsilon \circ \zeta)_\dagger \mathcal{O}_{\Gamma^*} \rightarrow (i_G^{Z^*})^+ (\varepsilon \circ \zeta)_+ \mathcal{O}_{\Gamma^*}\right) \\ &\simeq (i_G^{Z^*})^+ \text{im}((\varepsilon \circ \zeta)_\dagger \mathcal{O}_{\Gamma^*} \rightarrow (\varepsilon \circ \zeta)_+ \mathcal{O}_{\Gamma^*}) \\ &\simeq (i_G^{Z^*})^+ \mathcal{M}^{IC}(Z_X^*), \end{aligned}$$

where the second isomorphism follows from $(i_G^{\Gamma^*})^+ \mathcal{O}_{\Gamma^*} \simeq \mathcal{O}_G$, the fact that \mathcal{O}_{Γ^*} is non-characteristic for $i_G^{\Gamma^*}$ and [HTT08, Theorem 2.7.1(ii)]. The third isomorphism follows by base change and the fourth isomorphism follows from the fact that $i_G^{Z^*}$ is non-characteristic with respect to $(\varepsilon \circ \zeta)_\dagger \mathcal{O}_{\Gamma^*}$ and $(\varepsilon \circ \zeta)_+ \mathcal{O}_{\Gamma^*}$.

For the last claim consider the following diagram

$$\begin{array}{ccccc} Z & \xleftarrow{j_{Z^*}} & Z^* & \xleftarrow{i_G^{Z^*}} & \mathcal{Z} \\ \pi_Z^2 \downarrow & & \downarrow \delta & & \downarrow \alpha \\ V & \xleftarrow{j_{V^*}} & V^* & \xleftarrow{i_G^{V^*}} & \mathbb{C} \times \mathcal{KM} \end{array}$$

We have the following isomorphisms

$$\begin{aligned} \alpha_+ \mathcal{M}^{IC}(\mathcal{Z}_X) &\simeq \alpha_+ (i_G^{Z^*})^+ \mathcal{M}^{IC}(Z_X^*) \\ &\simeq \alpha_+ (i_G^{Z^*})^+ j_{Z^*}^+ \mathcal{M}^{IC}(Z_X) \\ &\simeq (i_G^{V^*})^+ j_{V^*}^+ \pi_{2+}^Z \mathcal{M}^{IC}(Z_X) \\ &\simeq i_{\mathcal{KM}}^+ \pi_{2+}^Z \mathcal{M}^{IC}(Z_X) \\ &\simeq i_{\mathcal{KM}}^+ \pi_{2+}^Z (\pi_1^Z)^+ \mathcal{M}^{IC}(X) \\ &\simeq i_{\mathcal{KM}}^+ \mathcal{R}(\mathcal{M}^{IC}(X)). \end{aligned}$$

The non-characteristic property of $i_{\mathcal{KM}} = j_{V^*} \circ i_G^{V^*}$ follows from Lemma 2.21 and the fact that $j_{V^*}^+ \mathcal{R}(\mathcal{M}^{IC}(X))$ is quasi- G -equivariant. \square

3 Fourier transformation and lattices

In this section we apply the Fourier transformation functor FL_W to the various \mathcal{D} -modules considered in section 2. For the families of Laurent polynomials resp. compactifications thereof that appear in mirror symmetry, we obtain \mathcal{D} -modules that can eventually be matched with the differential systems defined by quantum cohomology. They have in general irregular singularities, and this is reflected in the fact that although the modules considered in section 2 were monodromic on V , they do not have necessarily that property with respect to the vector bundle $V = \mathbb{C}_{\lambda_0} \times W \rightarrow W$. Hence the functor FL_W will in general not preserve regularity.

In the second part of this section, we study a lattice in the Fourier transformation of the Gauß-Manin system of the family of Laurent polynomials φ_B . It is given by a so-called twisted de Rham complex, however, in order to obtain a good hypergeometric description of it, we have to introduce a certain intermediate compactification of φ_B and replace this de Rham complex by a logarithmic version. Moreover, the parameters of the family φ_B have to be restricted to a Zariski open set excluding certain (but not all) singularities at infinity. Then we can show the necessary finiteness and freeness of the lattice. It will later correspond to the twisted quantum \mathcal{D} -module (see section 4), seen as a family of algebraic vector bundles over \mathbb{C}_z (not only over \mathbb{C}_z^*) with connection operator which is meromorphic along $\{z = 0\}$.

3.1 Localized Fourier-Laplace Transform

We discuss here a partial localized Fourier-Laplace transform of the Gauß-Manin systems of φ_B and of the \mathcal{D} -module $\mathcal{M}^{IC}(X^\circ, \mathcal{L})$.

Consider the product decomposition $V = \mathbb{C}_{\lambda_0} \times W$, where W is the hyperplane given by $\lambda_0 = 0$. We interpret V as a rank one bundle with base W and consider the Fourier-Laplace transformation with respect to the base W as in Definition 2.4, where we denote the coordinate on the dual fiber by τ . Set $z = 1/\tau$ and denote by $j_\tau : \mathbb{C}_\tau^* \times W \hookrightarrow \mathbb{C}_\tau \times W$ and $j_z : \mathbb{C}_\tau^* \times W \hookrightarrow \hat{V} := \mathbb{C}_z \times W = \mathbb{P}_\tau^1 \setminus \{\tau = 0\} \times W$ the canonical embeddings. Let \mathcal{N} be a \mathcal{D}_V -module, the partial, localized Fourier-Laplace transformation is defined by

$$\mathrm{FL}_W^{loc}(\mathcal{N}) := j_{z+} j_\tau^+ \mathrm{FL}_W(\mathcal{N}).$$

The localized Fourier-Laplace transformations of the Gauß-Manin systems are denoted by

$$\mathcal{G}^+ := \mathrm{FL}_W^{loc}(\mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W})), \quad (26)$$

$$\mathcal{G}^\dagger := \mathrm{FL}_W^{loc}(\mathcal{H}^0(\varphi_{B,\dagger} \mathcal{O}_{S \times W})). \quad (27)$$

We also consider the partial, localized Fourier-Laplace transform of the \mathcal{D} -modules $\mathcal{M}_B^{\tilde{\beta}}$. The following notation will be useful.

Definition 3.1. Let $\widehat{M}_B^{(\beta_0, \beta)}$ be the $D_{\widehat{V}}$ -module $D_{\widehat{V}}[z^{-1}]/I$, where I is the left ideal generated by the operators $\widehat{\square}_l, \widehat{E}_k - \beta_k z$ and $\widehat{E} - \beta_0 z$, which are defined by

$$\widehat{\square}_l := \prod_{i:l_i < 0} (z \cdot \partial_{\lambda_i})^{-l_i} - \prod_{i:l_i > 0} (z \cdot \partial_{\lambda_i})^{l_i},$$

$$\widehat{E} := z^2 \partial_z + \sum_{i=1}^t z \lambda_i \partial_{\lambda_i},$$

$$\widehat{E}_k := \sum_{i=1}^t b_{ki} z \lambda_i \partial_{\lambda_i}.$$

We denote the corresponding $\mathcal{D}_{\widehat{V}}$ -module by $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$.

Lemma 3.2. We have the following isomorphism

$$\mathrm{FL}_W^{loc}(\mathcal{M}_B^{\tilde{\beta}}) \simeq \widehat{\mathcal{M}}_B^{(\beta_0+1, \beta)}$$

for every $\tilde{\beta} = (\beta_0, \beta) \in \mathbb{Z}^{s+1}$.

Proof. This is an easy calculation, using the substitution

$$\lambda_0 \rightarrow -\partial_\tau = z^2 \partial_z \quad \text{and} \quad \partial_{\lambda_0} \rightarrow \tau = 1/z$$

and the fact that $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ is localized along $z = 0$. \square

Notice that in the lemma above we used the subscript \tilde{B} for the GKZ-system on the left hand side and the subscript B for its localized Fourier-Laplace transform on the right hand side. This notation takes into account the fact that the properties of the system $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ are governed by the geometry of the semi-group $\mathbb{N}\tilde{B}$, whereas the properties of its localized Fourier-Laplace transform $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ depend on the geometry of $\mathbb{N}B$. This explains the different sets of allowed parameters in Proposition 3.3 resp. Theorem 3.6 in contrast to Theorem 2.11 resp. Theorem 2.16 and Proposition 2.17.

The following proposition gives an isomorphism between the localized partial Fourier-Laplace transform of the Gauß-Manin systems \mathcal{G}^+ and \mathcal{G}^\dagger and the hypergeometric systems $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ introduced above.

Proposition 3.3. *There exists an $\delta_B \in \mathbb{N}B$ such that we have an isomorphism*

$$\mathcal{G}^+ \simeq \widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$$

for every $\beta_0 \in \mathbb{Z}$ and $\beta \in \delta_B + (\mathbb{R}_+ B \cap \mathbb{Z}^s)$. If $\mathbb{N}B$ is saturated, then δ_B can be taken to be $\underline{0} \in \mathbb{N}B$ (in particular, the statement holds for $(\beta_0, \beta) = (\beta_0, \underline{0}) \in \mathbb{Z}^{1+s}$).

Furthermore, we have an isomorphism

$$\mathcal{G}^\dagger \simeq \widehat{\mathcal{M}}_B^{(\beta'_0, -\beta')}$$

for every $\beta'_0 \in \mathbb{Z}$ and $\beta' \in (\mathbb{R}_+ B)^\circ \cap \mathbb{Z}^s$.

Proof. We construct the isomorphisms by applying the Fourier-Laplace transform FL_W to the exact sequences in Theorem 2.11. First notice that the first and last term in the exact sequences are free \mathcal{O}_V -modules, thus their Fourier-Laplace transform has support on $\tau = 0$, i.e. their localized Fourier-Laplace transform is 0. Thus there is some $\delta_{\tilde{B}} \in \mathbb{N}\tilde{B}$ such that we have the following isomorphisms

$$\mathcal{G}^+ = \text{FL}_W^{\text{loc}}(\mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W})) \simeq \text{FL}_W^{\text{loc}}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}})$$

and

$$\mathcal{G}^\dagger = \text{FL}_W^{\text{loc}}(\mathcal{H}^0(\varphi_{B,\dagger} \mathcal{O}_{S \times W})) \simeq \text{FL}_W^{\text{loc}}(\mathcal{M}_{\tilde{B}}^{-\tilde{\beta}'})$$

for any $\tilde{\beta} \in \delta_{\tilde{B}} + (\mathbb{R}_+ \tilde{B} \cap \mathbb{Z}^{s+1})$ and any $\tilde{\beta}' \in (\mathbb{R}_+ \tilde{B})^\circ \cap \mathbb{Z}^{s+1}$. Write $\delta_{\tilde{B}} = (\delta_0, \delta_B)$ with $\delta_B \in \mathbb{Z}^s$. Now given any $(\beta_0, \beta) \in \mathbb{Z} \times (\delta_B + (\mathbb{R}_+ B \cap \mathbb{Z}^s))$ resp. $(\beta'_0, \beta') \in \mathbb{Z} \times ((\mathbb{R}_+ B)^\circ \cap \mathbb{Z}^s)$ we can find a $\gamma_0, \gamma'_0 \in \mathbb{Z}$ such that $(\gamma_0, \beta) \in \delta_{\tilde{B}} + (\mathbb{R}_+ \tilde{B} \cap \mathbb{Z}^{s+1})$ resp. $(\gamma'_0, \beta') \in (\mathbb{R}_+ \tilde{B})^\circ \cap \mathbb{Z}^{s+1}$. It remains to show that there are isomorphism

$$\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} \simeq \widehat{\mathcal{M}}_B^{(\gamma_0, \beta)} \tag{28}$$

for $(\beta_0, \beta) \in \mathbb{Z} \times (\delta_B + (\mathbb{R}_+ B \cap \mathbb{Z}^s))$ and $(\gamma_0, \beta) \in \delta_{\tilde{B}} + (\mathbb{R}_+ \tilde{B} \cap \mathbb{Z}^{s+1})$ resp.

$$\widehat{\mathcal{M}}_B^{(\beta'_0, -\beta')} \simeq \widehat{\mathcal{M}}_B^{(-\gamma'_0, -\beta')} \tag{29}$$

for $(\beta'_0, \beta') \in \mathbb{Z} \times ((\mathbb{R}_+ B)^\circ \cap \mathbb{Z}^s)$ and $(-\gamma'_0, -\beta') \in ((\mathbb{R}_+ \tilde{B})^\circ \cap \mathbb{Z}^{s+1})$. Notice that $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ is localized along $z = 0$ for all $(\beta_0, \beta) \in \mathbb{Z}^{s+1}$ by Lemma (3.2). Therefore the morphism given by right multiplication with z

$$\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} \xrightarrow{\cdot z} \widehat{\mathcal{M}}_B^{(\beta_0-1, \beta)} \tag{30}$$

is an isomorphism, which shows (28) and (29).

Concerning the last statement, suppose that $\mathbb{N}B$ is saturated. Let $\beta \in \mathbb{N}B = (\mathbb{R}_+ B \cap \mathbb{Z}^s)$ and $\beta_0 \in \mathbb{Z}$ arbitrary. By [Rei12, Lemma 1.17] we have $\beta \notin s\text{Res}(B)$, where $s\text{Res}(B) \subset \mathbb{C}^s$ is the set of strongly resonant values (cf. [SW09, Definition 3.4]). Using [Rei12, Lemma 1.19] there exists a $\gamma_0 \in \mathbb{Z}$ such that $(\gamma_0, \beta) \notin s\text{Res}(\tilde{B})$. Now we argue as above, i.e. by [Rei12, Theorem 2.7] we have $\mathcal{G}^+ = \text{FL}_W^{\text{loc}}(\mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W})) \simeq \text{FL}_W^{\text{loc}}(\mathcal{M}_{\tilde{B}}^{(\gamma_0, \beta)})$ which in turn is isomorphic to $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$. \square

If the semigroup $\mathbb{N}B$ is saturated, we will compute the isomorphism above explicitly for $(\beta_0, \beta) = (0, \underline{0})$. For this we will need a direct description of the localized, partial Fourier-Laplace transformed Gauß-Manin system \mathcal{G}^+ .

Lemma 3.4. *Write $\varphi = (F, \pi)$, where $F : S \times W \rightarrow \mathbb{C}$, $(y, \underline{\lambda}) \mapsto -\sum_{i=1}^t \lambda_i y^{b_i}$ and $\pi : S \times W \rightarrow W$ is the projection. Recall from formula (26) that we denote by $\widehat{\mathcal{G}}^+$ the localized Fourier-Laplace transformation of the Gauß-Manin system of the morphism φ . Write $G^+ := H^0(\widehat{V}, \mathcal{G}^+)$ for its module of global sections. Then there is an isomorphism of $\mathcal{D}_{\widehat{V}}$ -modules*

$$G^+ \cong H^0 \left(\Omega_{S \times W/W}^{\bullet+s}[z^{\pm}], d - z^{-1} \cdot d_y F \wedge \right),$$

where d is the differential in the relative de Rham complex $\Omega_{S \times W/W}^{\bullet}$. The structure of a $\mathcal{D}_{\widehat{V}}$ -module on the right hand side is defined as follows

$$\begin{aligned} \partial_z(\omega \cdot z^i) &= i \cdot \omega \cdot z^{i-1} + F \cdot \omega \cdot z^{i-2}, \\ \partial_{\lambda_i}(\omega \cdot z^i) &:= \partial_{\lambda_i}(\omega) \cdot z^i - \partial_{\lambda_i} F \cdot \omega \cdot z^{i-1} = \partial_{\lambda_i}(\omega) \cdot z^i + \underline{y}^{b_i} \cdot \omega \cdot z^{i-1}, \end{aligned}$$

where $\omega \in \Omega_{S \times W/W}^r$.

Proof. The expression for the module G^+ as well as the formulas for the $\mathcal{D}_{\widehat{V}}$ -structure are an immediate consequence of the definition of the direct image functor. See, e.g. [Rei12, equations 2.0.18, 2.0.19], from which the desired formulas can be easily obtained. \square

Using the description of G^+ via relative differential forms, we find a distinguished element, which is (the class of) the volume form on S , that is

$$\omega_0 := \frac{dy_1}{y_1} \wedge \dots \wedge \frac{dy_s}{y_s}.$$

In the next lemma we compute the image of ω_0 under the isomorphisms in Proposition 3.3 under the assumption of normality of $\mathbb{N}B$.

Lemma 3.5. *Let $\mathbb{N}B$ be a saturated semigroup, then the isomorphism from Proposition 3.3*

$$\Phi : \mathcal{G}^+ \xrightarrow{\cong} \widehat{\mathcal{M}}_B^{(0,0)}$$

maps ω_0 to 1.

Proof. Recall from the proof of Proposition 3.3, that there exists a $\gamma_0 \in \mathbb{Z}$ such that $(\gamma_0, \underline{0}) \notin sRes(\widetilde{B})$ (notice that here we only assume that $\mathbb{N}B$ is saturated which does not imply that $\mathbb{N}\widetilde{B}$ is saturated). Denote by

$$\psi_{(\gamma_0, \underline{0})} : \Gamma(V, \mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W})) \rightarrow M_B^{(\gamma_0, \underline{0})}$$

the morphism from Theorem 2.11. We first compute the image of ω_0 under the morphism $\psi_{(\gamma_0, \underline{0})}$ using the description of $\mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W})$ by relative differential forms (see e.g. [Rei12, Equation 2.0.17]). We will use the following two facts of loc. cit. Proposition 2.8 whose proofs extend directly to our slightly more general situation (there it was assumed that $\mathbb{N}\widetilde{B}$ is saturated). Namely first, that there exists a non-zero morphism $M_B^{(-1, \underline{0})} \rightarrow \Gamma(V, \mathcal{H}^0(\varphi_{B,+} \mathcal{O}_{S \times W}))$ which sends 1 to ω_0 and second that $\psi_{(\gamma_0, \underline{0})}(\omega_0) \neq 0$. Concatenating this morphism with $\psi_{(\gamma_0, \underline{0})}$ gives a non-zero morphism $M_B^{(-1, \underline{0})} \rightarrow M_B^{(\gamma_0, \underline{0})}$, where $1 \in M_B^{(-1, \underline{0})}$ is sent to the image of ω_0 under $\psi_{(\gamma_0, \underline{0})}$. By [Rei12, Proposition 1.24] this morphism is uniquely given by right multiplication with $\partial_{\lambda_0}^{\gamma_0+1}$ (up to a non-zero constant). Applying now the partial localized Fourier-Laplace transform to the morphism $\psi_{(\gamma_0, \underline{0})}$, we see that $\psi_{(\gamma_0, \underline{0})}(\omega_0) = z^{-\gamma_0-1}$. Using the isomorphism $\widehat{\mathcal{M}}_B^{(\gamma_0, \underline{0})} \xrightarrow{\cong} \widehat{\mathcal{M}}_B^{(\gamma_0'-1, \underline{0})}$, which holds for any $\gamma_0' \in \mathbb{Z}$, shows the claim. \square

By Proposition 2.16, we can now give a concrete description of the partial, localized Fourier-Laplace transform $\widehat{\mathcal{M}}^{IC}(X^\circ, \mathcal{L}) := \mathrm{FL}_W^{loc}(\mathcal{M}^{IC}(X^\circ, \mathcal{L}))$ of the intersection cohomology \mathcal{D} -module $\mathcal{M}^{IC}(X^\circ, \mathcal{L})$.

Theorem 3.6. *Let $\beta \in \delta_B + (\mathbb{R}_+ B \cap \mathbb{Z}^s)$, $\beta' \in (\mathbb{R}_+ B)^\circ \cap \mathbb{Z}^s$ and $\beta_0, \beta'_0 \in \mathbb{Z}$, then we have the following isomorphisms*

$$\widehat{\mathcal{M}}^{IC}(X^\circ, \mathcal{L}) \simeq \mathrm{im} \left(\widehat{\mathcal{M}}_B^{(\beta'_0, -\beta')} \xrightarrow{\cdot z^{\beta'_0 - \beta_0} \partial^{\beta + \beta'}} \widehat{\mathcal{M}}_B^{(\beta_0, \beta)} \right),$$

resp.

$$\widehat{\mathcal{M}}^{IC}(X^\circ, \mathcal{L}) \simeq \widehat{\mathcal{M}}_B^{(\beta'_0, -\beta')} / \widehat{\Gamma}_\partial \left(\widehat{\mathcal{M}}_B^{(\beta'_0, -\beta')} \right),$$

where $\widehat{\Gamma}_\partial \left(\widehat{\mathcal{M}}_B^{(\beta'_0, -\beta')} \right)$ is the sub- \mathcal{D} -module corresponding to the sub- \mathcal{D} -module

$$\widehat{\Gamma}_\partial \left(\widehat{\mathcal{M}}_B^{(\beta'_0, -\beta')} \right) := \{m \in \widehat{\mathcal{M}}_B^{(\beta'_0, -\beta')} \mid \exists n \in \mathbb{N} \text{ with } (\partial^{\beta + \beta'})^n \cdot m = 0\}.$$

Furthermore, if $\mathbb{N}B$ is saturated, then δ_B can be taken to be $\underline{0} \in \mathbb{N}B$ (so that, similarly to Proposition 3.3, the statement holds true for $(\beta_0, \beta) = (\beta_0, \underline{0}) \in \mathbb{Z}^{1+s}$).

Proof. Using the isomorphism

$$\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} \xrightarrow{\cdot z} \widehat{\mathcal{M}}_B^{(\beta_0 - 1, \beta)}, \quad (31)$$

which holds for every $(\beta_0, \beta) \in \mathbb{Z}^{s+1}$, we can assume that $(\beta_0 + 1, \beta) \in \delta_{\widetilde{B}} + (\mathbb{R}_+ \widetilde{B} \cap \mathbb{Z}^{s+1})$ resp. $(\beta'_0 + 1, \beta') \in (\mathbb{R}_+ \widetilde{B})^\circ \cap \mathbb{Z}^{s+1}$. Then the first isomorphism follows by applying the functor FL_W^{loc} to the isomorphism in Theorem 2.16 and Lemma 3.2.

For the second isomorphism we can assume again that $(\beta'_0 + 1, \beta') \in (\mathbb{R}_+ \widetilde{B})^\circ \cap \mathbb{Z}^{s+1}$. Now the desired statement is obtained by applying FL_W^{loc} to the second isomorphism in Proposition 2.17 and the fact that $\widehat{\Gamma}_\partial(\widehat{\mathcal{M}}_B^{(\beta'_0, -\beta')})$ is stable under left multiplication with z .

Now assume that $\mathbb{N}B$ is saturated and let $\beta \in \mathbb{N}B$. Arguing as in the last part of the proof of Proposition 3.3 we can find a $\gamma_0 \in \mathbb{Z}$ such that $(\gamma_0, \beta) \notin s\mathrm{Res}(\beta)$. By [SW09, Corollary 3.7] we have an isomorphism $\mathrm{FL}(h_+ \mathcal{O}_T) \simeq \mathcal{M}_{\widetilde{B}}^{(\gamma_0, \beta)}$. Now the proof of Theorem 2.16 shows that

$$\mathcal{M}^{IC}(X^\circ, \mathcal{L}) \simeq \mathrm{im}(\mathcal{M}_{\widetilde{B}}^{-\beta'} \xrightarrow{\cdot \partial^{(\gamma_0, \beta) + \beta'}} \mathcal{M}_{\widetilde{B}}^{(\gamma_0, \beta)}).$$

Now applying the functor F_W^{loc} and using the isomorphism (31) shows the claim in the saturated case. \square

3.2 Tameness and Lattices

In this section we define a natural lattice in the Fourier-Laplace transformed Gauß-Manin system \mathcal{G}^+ outside some bad locus where the Laurent polynomial acquires singularities at infinity. For this we need to study the characteristic variety of the Gauß-Manin system of φ_B and the corresponding GKZ system $\mathcal{M}_{\widetilde{B}}^{\beta}$. Throughout this section we assume that $\mathbb{N}B$ is a saturated semigroup. Recall the embedding of the torus S in the projective space from formula (4)

$$S \xrightarrow{j} X \xrightarrow{i} \mathbb{P}(V').$$

The projective variety X serves as a convenient ambient space to compactify fibers of the family of Laurent polynomials φ_B . Let X^{aff} be the restriction of X to the affine chart of $\mathbb{P}(V')$ given by $\mu_0 = 1$. The affine variety X^{aff} is therefore the closure of the map

$$g_B : S \longrightarrow \mathbb{C}^t, \\ (y_1, \dots, y_s) \mapsto (\underline{y}^{\underline{b}_1}, \dots, \underline{y}^{\underline{b}_t}). \quad (32)$$

Hence X^{aff} is isomorphic to $\text{Spec}(\mathbb{C}[\mathbb{N}B])$. Consider the following diagram, which is a refinement of a part of diagram (19):

$$\begin{array}{ccccccc}
\Gamma & \xrightarrow{\theta_2} & Z_{X^{aff}} & \xrightarrow{\theta_1} & Z_X & \xrightarrow{\eta} & Z & \xrightarrow{\pi_2^Z} & V \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \pi_1^Z & & \\
S & \xrightarrow{j_2} & X^{aff} & \xrightarrow{j_1} & X & \xrightarrow{i} & \mathbb{P}(V') & &
\end{array} \tag{33}$$

where j_1 and j_2 are the canonical inclusions and the three squares are cartesian. Recall that $Z \subset \mathbb{P}(V') \times V$ was given by the incidence relation $\sum_{i=0}^t \lambda_i \mu_i = 0$ and the composed map $g = i \circ j = i \circ j_1 \circ j_2$ was defined by formula (3). Thus Γ resp. $Z_{X^{aff}}$ is the subvariety of $S \times V = S \times \mathbb{C}_{\lambda_0} \times W$ resp. $X^{aff} \times V$ given by the equation $\lambda_0 + \sum_{i=1}^t \lambda_i \underline{y}^{b_i} = 0$. It follows from the definition that Γ is the graph of φ_B . Therefore the maps

$$\pi_{Z_X} := \pi_2^Z \circ \eta : Z_X \longrightarrow V$$

resp.

$$\pi_{Z_{X^{aff}}} := \pi_2^Z \circ \eta \circ \theta_1 : Z_{X^{aff}} \longrightarrow V$$

provide natural (partial) compactifications of the family of Laurent polynomials φ_B . Putting $H_{\tilde{\lambda}} := \{\sum_{i=0}^t \lambda_i \mu_i = 0\} \subset \mathbb{P}(V')$ for any $\tilde{\lambda} \in V$, we see that the fiber $\pi_{Z_X}^{-1}(\tilde{\lambda})$ resp. $\pi_{Z_{X^{aff}}}^{-1}(\tilde{\lambda})$ is given by $X \cap H_{\tilde{\lambda}}$ resp. $\{\lambda_0 + \sum_{i=1}^t \lambda_i \underline{y}^{b_i} = 0\} \subset X^{aff}$.

Recall that the toric variety X has a natural stratification by torus orbits $X^0(\Gamma)$, which are in one-to-one correspondence with the faces Γ of the polytope Q , which is the convex hull of the elements $\{\underline{b}_0 := \underline{0}, \underline{b}_1, \dots, \underline{b}_t\}$. Notice that the stratification $\mathcal{S} := \{X^0(\Gamma)\}$ is a Whitney stratification of X (see e.g. [Dim92, Proposition 1.14]).

By [GKZ08, Chapter 5, Prop 1.9] the orbit $X^0(\Gamma) \simeq (\mathbb{C}^*)^{\dim(\Gamma)}$ is the image of the map

$$\begin{aligned}
g_\Gamma : S &\longrightarrow \mathbb{P}(V'), \\
(y_1, \dots, y_s) &\mapsto (\varepsilon_0 1 : \varepsilon_1 \underline{y}^{b_1} : \dots : \varepsilon_t \underline{y}^{b_t}),
\end{aligned}$$

where $\varepsilon_i = 0$ if $\underline{b}_i \notin \Gamma$ and $\varepsilon_i = 1$ if $\underline{b}_i \in \Gamma$. It is easy to see that

$$X^{aff} = \bigcup_{\Gamma \mid 0 \in \Gamma} X^0(\Gamma)$$

and this induces a Whitney stratification of X^{aff} .

The preimage of $X^0(\Gamma) \cap H_{\tilde{\lambda}}$ under g_Γ is given by

$$\{(y_1, \dots, y_s) \in S \mid \sum_{\underline{b}_i \in \Gamma} \lambda_i \underline{y}^{b_i} = 0\}.$$

It follows from [GKZ08, Chapter 5.D] that the morphism $p_\Gamma : S \longrightarrow X^0(\Gamma) \simeq (\mathbb{C}^*)^{\dim(\Gamma)}$ is a trivial fibration with fiber being isomorphic to $(\mathbb{C}^*)^{d-\dim(\Gamma)}$.

Denote by $S_\Gamma^{crit, \tilde{\lambda}}$ the set

$$\left\{ (y_1, \dots, y_s) \in S \mid \sum_{\underline{b}_i \in \Gamma} \lambda_i \underline{y}^{b_i} = 0; \quad y_k \partial_{y_k} \left(\sum_{\underline{b}_i \in \Gamma} \lambda_i \underline{y}^{b_i} \right) = 0 \quad \text{for all } k \in \{1, \dots, s\} \right\}. \tag{34}$$

Then its image under p_Γ is exactly the singular set $\text{sing}(X^0(\Gamma) \cap H_{\tilde{\lambda}})$ of $X^0(\Gamma) \cap H_{\tilde{\lambda}}$. This motivates the following definition.

Definition 3.7. Let $\tilde{\lambda} \in V$

1. The fiber $\pi_{Z^X}^{-1}(\tilde{\lambda})$ has stratified singularities in $X^0(\Gamma)$ if $X^0(\Gamma) \cap H_{\tilde{\lambda}}$ is singular, i.e. $S_{\Gamma}^{crit, \tilde{\lambda}} \neq \emptyset$.
2. The set

$$\begin{aligned} \Delta_B &:= \{\tilde{\lambda} \in V \mid S_{\Gamma}^{crit, \tilde{\lambda}} \neq \emptyset\} \\ &= \{\tilde{\lambda} \in V \mid \varphi_B^{-1}(\tilde{\lambda}) \text{ is singular}\} \end{aligned}$$

is called the discriminant of φ_B .

3. The fiber $\varphi_B^{-1}(\tilde{\lambda})$ has **singularities at infinity** if there exists a proper face Γ of the Newton polyhedron Q so that $S_{\Gamma}^{crit, \tilde{\lambda}} \neq \emptyset$. The set

$$\Delta_B^{\infty} := \{\tilde{\lambda} \in V \mid \exists \Gamma \neq Q \text{ so that } S_{\Gamma}^{crit, \tilde{\lambda}} \neq \emptyset\}$$

is called the non-tame locus of φ_B .

4. The fiber $\varphi_B^{-1}(\tilde{\lambda})$ has **bad singularities at infinity** if there exists a proper face Γ of the Newton polyhedron Q not containing the origin such that $S_{\Gamma}^{crit, \tilde{\lambda}} \neq \emptyset$. The set

$$\Delta_B^{bad} := \{\tilde{\lambda} \in V \mid \exists \Gamma \neq Q, 0 \notin \Gamma \text{ so that } S_{\Gamma}^{crit, \tilde{\lambda}} \neq \emptyset\} \subset \Delta_B^{\infty}$$

is called the bad locus of φ_B .

Remark 3.8. Notice that Δ_B^{bad} is independent of λ_0 . We denote its projection to W by W^{bad} . Let $W^* = W \setminus \{\lambda_1 \dots \lambda_t = 0\}$ and define

$$W^{\circ} := W^* \setminus W^{bad},$$

which we call the set of good parameters for φ_B .

Recall that X^{aff} is isomorphic to $\text{Spec}(R_B)$ with $R_B := \mathbb{C}[\mathbb{N}B]$. Let $\underline{\lambda} \in W$ and set $f_{\underline{\lambda}}(\bullet) := \varphi_B(\bullet, \underline{\lambda})$. Notice that the Laurent polynomials $f_{\underline{\lambda}}$ and $y_k \partial f_{\underline{\lambda}} / \partial y_k$ for $k = 1, \dots, s$, which were defined on S before are actually elements of R_B and can thus naturally be considered as functions on X^{aff} .

Lemma 3.9. Let $\underline{\lambda} \in W^{\circ}$ be a good parameter, then

$$\dim_{\mathbb{C}} (R_B / (y_k \partial f_{\underline{\lambda}} / \partial y_k)_{k=1, \dots, s}) = \text{vol}(Q),$$

where the volume of a hypercube $[0, 1]^s \subset \mathbb{R}^s$ is normalized to $s!$. Moreover, we have

$$\text{supp}(R_B / (y_k \partial f_{\underline{\lambda}} / \partial y_k)_{k=1, \dots, s}) = \bigcup_{\lambda_0 \in \mathbb{C}} \text{sing}_{\mathcal{S}}(\pi_{Z^X}^{-1}(\lambda_0, \underline{\lambda})),$$

where we see $\pi_{Z^X}^{-1}(\lambda_0, \underline{\lambda})$ as a subset of $X \subset \mathbb{P}(V')$ and where $\text{sing}_{\mathcal{S}}(\pi_{Z^X}^{-1}(\lambda_0, \underline{\lambda}))$ denotes the stratified singular locus with respect to the stratification \mathcal{S} of X by torus orbits defined above.

Proof. For the first claim consider the following increasing filtration on R_B . Let as above Q be the convex hull of $\underline{b}_1, \dots, \underline{b}_t$ and 0 in \mathbb{R}^s . Let $u \in \mathbb{N}B$ then the weight of \underline{y}^u is defined by $\inf\{\lambda \in \mathbb{R}_+ \mid u \in \lambda \cdot Q\}$. It is easy to see that there is an integer e so that all weights lie in $e^{-1}\mathbb{N}$. Denote by $R_B^{\frac{k}{e}}$ the elements in R_B with weight $\leq k/e$. Let $gr R_B$ the graduated ring with respect to this filtration. By [Ado94, Equation 5.12] we have

$$\dim_{\mathbb{C}} gr(R_B) / \overline{(y_k \partial f_{\underline{\lambda}} / \partial y_k)_{k=1, \dots, s}} = \text{vol}(Q),$$

where $\overline{(y_k \partial f_{\underline{\lambda}} / \partial y_k)_{k=1, \dots, s}}$ is the image of $(y_k \partial f_{\underline{\lambda}} / \partial y_k)_{k=1, \dots, s}$ in $gr(R_B)$. It remains to show that

$$\dim_{\mathbb{C}} gr(R_B) / \overline{(y_k \partial f_{\underline{\lambda}} / \partial y_k)_{k=1, \dots, s}} = \dim_{\mathbb{C}} R_B / (y_k \partial f_{\underline{\lambda}} / \partial y_k)_{k=1, \dots, s}.$$

The proof of this equality is an easy adaptation of the proof of [Ado94, Theorem 5.4].

For the proof of the second statement we notice first that

$$\text{sing}_S(\pi_{Z_X}^{-1}(\lambda_0, \underline{\lambda})) = \bigcup_{\Gamma|0 \in \Gamma} \text{sing}(X^0(\Gamma) \cap H_{(\lambda_0, \underline{\lambda})})$$

because the fiber over $(\lambda_0, \underline{\lambda})$ has no bad singularities at infinity.

Define the following r hyperplanes $H_{\underline{\lambda}}^k$ for $k \in \{1, \dots, s\}$ and $\underline{\lambda} \in W^\circ$:

$$H_{\underline{\lambda}}^k := \{(\mu_0 : \dots : \mu_t) \in \mathbb{P}(V') \mid \sum_{i=1}^t b_{ki} \lambda_i \mu_i = 0\}.$$

We have $\text{sing}(X^0(\Gamma) \cap H_{(\lambda_0, \underline{\lambda})}) = X^0(\Gamma) \cap H_{(\lambda_0, \underline{\lambda})} \cap (\bigcap_{k=1}^s H_{\underline{\lambda}}^k)$ by equation (34) and therefore

$$\text{sing}_S(\pi_{Z_X}^{-1}(\lambda_0, \underline{\lambda})) = X^{\text{aff}} \cap H_{(\lambda_0, \underline{\lambda})} \cap \left(\bigcap_{k=1}^s H_{\underline{\lambda}}^k \right).$$

Notice that

$$\begin{aligned} \bigcup_{\lambda_0 \in \mathbb{C}} (X^{\text{aff}} \cap H_{(\lambda_0, \underline{\lambda})} \cap \left(\bigcap_{k=1}^s H_{\underline{\lambda}}^k \right)) &= \bigcup_{\lambda_0 \in \mathbb{C}} \text{supp}(R_B/R_B(f_{\underline{\lambda}} - \lambda_0) + R_B(\partial f_{\underline{\lambda}}/\partial y_k)_{k=1, \dots, s}) \\ &= \text{supp}(R_B/R_B(\partial f_{\underline{\lambda}}/\partial y_k)_{k=1, \dots, s}), \end{aligned}$$

which shows the claim. \square

Let \tilde{B} be the $(s+1) \times (t+1)$ -matrix as introduced before Definition 2.8. Let \tilde{Q} be the convex hull of $\tilde{b}_0, \dots, \tilde{b}_t$ in \mathbb{R}^{s+1} . Notice that $\tilde{Q} \subset \{1\} \times \mathbb{R}^s$ and therefore no face $\tilde{\Gamma}$ of \tilde{Q} contains the origin. Adolphson characterized the characteristic variety $\text{char}(\mathcal{M}_{\tilde{B}}^{\tilde{\beta}})$ of the GKZ system $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ as follows. Let $T^*V \simeq V \times V'$ be the holomorphic cotangent bundle with coordinates $(\lambda_0, \dots, \lambda_t, \mu_0, \dots, \mu_t)$. Define the following Laurent polynomials on $(\mathbb{C}^*)^{s+1}$

$$\begin{aligned} \tilde{f}_{\underline{\lambda}}(\underline{y}) &:= \tilde{f}_{\underline{\lambda}, \tilde{Q}}(\underline{y}) := \sum_{i=0}^t \lambda_i \underline{y}^{\tilde{b}_i}, \\ \tilde{f}_{\underline{\lambda}, \tilde{\Gamma}}(\underline{y}) &:= \sum_{\tilde{b}_i \in \tilde{\Gamma}} \lambda_i \underline{y}^{\tilde{b}_i}, \end{aligned}$$

where we define $\underline{y}^{\tilde{b}_i} := \prod_{k=0}^r y_k^{\tilde{b}_{ki}}$.

Lemma 3.10 ([Ado94] Lemma 3.2, Lemma 3.3).

1. For each $(\tilde{\lambda}^{(0)}, \tilde{\mu}^{(0)}) \in \text{char}(\mathcal{M}_{\tilde{A}}^{\tilde{\beta}})$ there exists a (possibly empty) face $\tilde{\Gamma}$ such that $\tilde{\mu}_j^{(0)} \neq 0$ if and only if $\tilde{b}_j \in \tilde{\Gamma}$.
2. If $\tilde{\lambda}^{(0)}$ is a singular point of $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ and $\tilde{\Gamma}$ the corresponding (non-empty) face, then the Laurent polynomials $\partial \tilde{f}_{\tilde{\lambda}^{(0)}, \tilde{\Gamma}} / \partial y_0, \dots, \partial \tilde{f}_{\tilde{\lambda}^{(0)}, \tilde{\Gamma}} / \partial y_s$ have a common zero in $(\mathbb{C}^*)^{s+1}$.

We can use this result in the next lemma to compute the singular locus of the \mathcal{D} -modules we are interested in.

Lemma 3.11. *The singular locus of $\mathcal{M}_{\tilde{B}}^{\tilde{\beta}}$ as well as the singular locus of the modules $\mathcal{H}^0(\varphi_{B+} \mathcal{O}_{S \times W})$ resp. $\mathcal{H}^0(\varphi_{B+} \mathcal{O}_{S \times W})$ is given by*

$$\Delta_S := \Delta_B \cup \Delta_B^\infty.$$

Proof. Notice that the polytope $\tilde{Q} \subset \{1\} \times \mathbb{R}^s$ is just the shifted polytope $Q \subset \mathbb{R}^s$ defined above. One easily sees that the Laurent polynomials $\partial \tilde{f}_{\tilde{\lambda}^{(0)}, \tilde{Q}} / \partial y_0, \dots, \partial \tilde{f}_{\tilde{\lambda}^{(0)}, \tilde{Q}} / \partial y_s$ have a common zero in $(\mathbb{C}^*)^{s+1}$ if and only if $\varphi_B^{-1}(\tilde{\lambda}^{(0)})$ is singular, i.e. the set of $\tilde{\lambda}^{(0)}$'s which satisfy this condition is exactly the discriminant Δ_B of φ_B . If there exists a proper face $\tilde{\Gamma}$ of \tilde{Q} such that the Laurent polynomials $\partial \tilde{f}_{\tilde{\lambda}^{(0)}, \tilde{\Gamma}} / \partial y_0, \dots, \partial \tilde{f}_{\tilde{\lambda}^{(0)}, \tilde{\Gamma}} / \partial y_s$ have a common zero in $(\mathbb{C}^*)^{s+1}$, then then fiber $\varphi_B^{-1}(\tilde{\lambda}^{(0)})$ has a singularity at infinity, i.e. its compactification has a singularity in $X^0(\Gamma)$, where Γ is the corresponding face of Q . \square

Lemma 3.12. *The restriction of the discriminant Δ_S to $\mathbb{C} \times W^\circ \subset V$ is finite over $W^\circ \subset W$.*

Proof. We will first show quasi-finiteness of the map $p : \Delta_{S|\mathbb{C} \times W^\circ} \rightarrow W^\circ$. First notice that we have $\Delta_{S|\mathbb{C} \times W^\circ} = (\Delta_S \setminus \Delta_B^{bad})|_{\mathbb{C} \times W^\circ}$. Fix some $\underline{\lambda} \in W^\circ$. We have to show that $\Delta_{S|\mathbb{C} \times \{\underline{\lambda}\}}$ is a finite set. By the definition of Δ_S it is enough to show that $\text{sing}_S(\pi_{Z_X}^{-1}(\lambda_0, \underline{\lambda}))$ is a finite set, but this is Lemma 3.9.

To prove finiteness of the map $p : \Delta_{S|\mathbb{C} \times W^\circ} \rightarrow W^\circ$ it remains to show that it is proper. Let K be any compact subset of W° . Suppose that $p^{-1}(K)$ is not compact, then it must be unbounded in $V \simeq \mathbb{C}^{t+1}$ for the standard metric. Hence there is a sequence $(\lambda_0^{(i)}, \underline{\lambda}^{(i)}) \in p^{-1}(K)$ with $\lim_{i \rightarrow \infty} |\lambda_0^{(i)}| = \infty$, as K is closed and bounded in $W^\circ \subset W = \mathbb{C}^t$.

In order to construct a contradiction, we use the partial compactification of the family φ_B from above. Recall the spaces $Z := \{\sum_{i=0}^t \lambda_i \cdot \mu_i = 0\} \subset \mathbb{P}(V') \times V$ and $Z_X := (X \times V) \cap Z$. Introduce the spaces $Z_k := \{\sum_{i=1}^t b_{ki} \lambda_i \mu_i = 0\}$ for $k \in \{1, \dots, t\}$. Then $Z_X \cap (\bigcap_{k=1}^d Z_k)$ is the stratified critical locus $\text{crit}_S(\pi_{Z_X})$ of the family π_{Z_X} , where we denote by abuse of notation by S also the stratification on Z_X induced from the torus stratification on X used above.

Because the projection from the stratified critical locus $\text{crit}_S(\pi_{Z_X})$ of π_{Z_X} to Δ_S is onto, there is a sequence $((\mu_0^{(i)} : \underline{\mu}^{(i)}), (\lambda_0^{(i)}, \underline{\lambda}^{(i)})) \in X^{aff} \times p^{-1}(K)$ projecting under $\pi_{Z_X|X^{aff} \times p^{-1}(K)}$ to $(\lambda_0^{(i)}, \underline{\lambda}^{(i)})$ (Notice that we consider here X^{aff} as a subset of $\mathbb{P}(V')$ under the embedding $i \circ j_1$). Consider the first component of the sequence $((\mu_0^{(i)} : \underline{\mu}^{(i)}), (\lambda_0^{(i)}, \underline{\lambda}^{(i)}))$, then this is a sequence $(\mu_0^{(i)} : \underline{\mu}^{(i)})$ in X which converges (after possibly passing to a subsequence) to a limit $(0 : \mu_1^{\text{lim}} : \dots : \mu_t^{\text{lim}})$ (this is forced by the incidence relation $\sum_{i=0}^t \lambda_i \mu_i$). In other words this limit lies in $X \setminus X^{aff}$ by the definition of X^{aff} before equation (32). But because $(X \times V) \cap Z \cap \bigcap_{k=1}^d Z_k = Z_X \cap \bigcap_{k=1}^d Z_k$ is closed, the point $((0 : \mu_1^{\text{lim}} : \dots : \mu_t^{\text{lim}}), (\lambda_0^{\text{lim}}, \underline{\lambda}^{\text{lim}}))$ lies in $((X \setminus X^{aff}) \times p^{-1}(K)) \cap Z \cap \bigcap_{k=1}^d Z_k$, i.e. $\pi_{Z_X}^{-1}(\lim_{i \rightarrow \infty} (\lambda_0^{(i)}, \underline{\lambda}^{(i)}))$ has a bad singularity at infinity, which is a contradiction by the definition of W° . \square

We can now prove the following regularity property of $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$, which is essentially the same proof as in [RS10, Lemma 4.4].

Lemma 3.13. *Consider $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ as a $\mathcal{D}_{\mathbb{P}^1 \times \overline{W}}$ -module, where \overline{W} is a smooth projective compactification of W . Then $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ is regular outside $(\{z = 0\} \times W) \cup (\mathbb{P}_z^1 \times (\underline{W} \setminus W^\circ))$ and smooth on $\mathbb{C}_z^* \times W^\circ$.*

Proof. It suffices to show that any $\underline{\lambda} = (\lambda_1, \dots, \lambda_t) \in W^\circ$ has a small analytic neighborhood $W_\underline{\lambda}^\circ \subset W^{\circ an}$ such that the partial analytization $\mathcal{O}_{W_\underline{\lambda}^\circ}^{an}[\tau, \tau^{-1}] \otimes_{\mathcal{O}_{\mathbb{C}_\tau^* \times W^\circ}} \widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ is regular on $\mathbb{C}_\tau \times W_\underline{\lambda}^\circ$ (but not at $\tau = \infty$). This is precisely the statement of [DS03, Theorem 1.11 (1)], taking into account the regularity of $\mathcal{M}_B^{\tilde{\beta}}$ (c.f. [Hot98, section 6]), the fact that the singular locus of $\mathcal{M}_B^{\tilde{\beta}}$ coincides with Δ_S (see Lemma 3.11) as well as the last lemma (notice that the non-characteristic assumption in [DS03, Theorem 1.11 (1)] is satisfied, see, e.g., [Pha79, page 281]). \square

The next step is to study several natural lattices in $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$. They are defined in terms of \mathcal{R} -modules, see the end of subsection 2.1.

Definition 3.14. 1. Consider the left ideal $\mathcal{I} := \mathcal{R}_{\mathbb{C}_z \times W^*}(\widehat{\square}_l)_{l \in \mathbb{L}} + \mathcal{R}_{\mathbb{C}_z \times W^*}(\widehat{E}_k - z \cdot \beta_k)_{k=1, \dots, r} + \mathcal{R}_{\mathbb{C}_z \times W^*}(\widehat{E} - z \cdot \beta_0)$ in $\mathcal{R}_{\mathbb{C}_z \times W^*}$ and write ${}^*\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ for the cyclic \mathcal{R} -module $\mathcal{R}_{\mathbb{C}_z \times W^*} / \mathcal{I}$. Here the operators $\widehat{\square}_l, \widehat{E}_k$ and \widehat{E} are those from Definition 3.1.

2. Consider the open inclusions $W^\circ \subset W^* \subset W$ and define the $\mathcal{D}_{\mathbb{C}_z \times W^\circ}$ -module

$$\circ\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} := \left(\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} \right)_{|\mathbb{C}_z \times W^\circ}$$

and the $\mathcal{R}_{\mathbb{C}_z \times W^\circ}$ -module

$$\circ\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} := \left({}^*\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} \right)_{|\mathbb{C}_z \times W^\circ}.$$

Remark 3.15.

1. We have $\mathcal{D}_{\mathbb{C}_z \times W^*} \otimes_{\mathcal{R}_{\mathbb{C}_z \times W^*}} \circ\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} = \widehat{\mathcal{M}}_B^{(\beta_0, \beta)}_{|\mathbb{C}_z \times W^*}$.
2. The restriction of $\circ\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ to $\mathbb{C}_z^* \times W^*$ equals the restriction of $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ to $\mathbb{C}_z^* \times W^*$.
3. For $z^2 \partial_z (\circ\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}) = \mathcal{R}'/\mathcal{I}'$, where \mathcal{I}' is given by

$$\mathcal{I}' := \mathcal{R}'(\widehat{\square}_l)_{l \in \mathbb{L}} + \mathcal{R}'(\widehat{E}_k - z \cdot \beta_k)_{k=1, \dots, r}.$$

Lemma 3.16. The quotient $\circ\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}/z \cdot \circ\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ is the sheaf of commutative \mathcal{O}_{W^*} -algebras associated to

$$\frac{\mathbb{C}[\lambda_1^\pm, \dots, \lambda_t^\pm, \kappa_1, \dots, \kappa_t]}{(\prod_{l_i < 0} \kappa_i^{-l_i} - \prod_{l_i > 0} \kappa_i^{l_i})_{l_i \in \mathbb{L}} + (\sum_{i=1}^t b_{ki} \lambda_i \kappa_i)_{k=1, \dots, s}} \simeq \frac{\mathbb{C}[\text{NB}][\lambda_1^\pm, \dots, \lambda_t^\pm]}{y_k \partial f_\lambda / \partial y_k}, \quad (35)$$

where $y_k \partial f_\lambda / \partial y_k = \sum_{i=1}^t b_{ki} \lambda_i y^{b_i}$.

Proof. Let κ_i be the class of $z \partial \lambda_i$. Because the commutator $[\kappa_i, \lambda_i]$ is zero we see that $\circ\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}/z \cdot \circ\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ is a commutative algebra and isomorphic to the module on the left hand side of equation (35). To show the isomorphism (35), consider the $\mathbb{C}[\lambda_1^\pm, \dots, \lambda_t^\pm]$ -linear morphism

$$\begin{aligned} \psi : \mathbb{C}[\lambda_1^\pm, \dots, \lambda_t^\pm, \kappa_1, \dots, \kappa_t] &\longrightarrow \mathbb{C}[\text{NB}][[\lambda_1^\pm, \dots, \lambda_t^\pm], \\ \kappa_i &\mapsto \underline{y}^{b_i} \end{aligned}$$

which is surjective by the definition of $\mathbb{C}[\text{NB}]$. The kernel of this map is equal to $(\prod_{l_i < 0} \kappa_i^{-l_i} - \prod_{l_i > 0} \kappa_i^{l_i})_{l_i \in \mathbb{L}}$ by [MS05, Theorem 7.3]. Finally notice that $\psi(\sum_{i=1}^t b_{ki} \lambda_i \kappa_i) = y_k \partial f_\lambda / \partial y_k$, which shows the claim. \square

We need the following result saying that the GKZ-system \mathcal{M}_B^β is isomorphic to the restriction of the Fourier-Laplace transformed GKZ system $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$.

Lemma 3.17. Let $i_1 : \{1\} \times W \longrightarrow \widehat{V} = \mathbb{C}_z \times W$ be the canonical inclusion. Then

$$\mathcal{H}^0 \left(i_1^+ \widehat{\mathcal{M}}_B^{(\beta_0, \beta)} \right) \simeq \mathcal{M}_B^\beta.$$

Proof. During the proof we will work with modules of global sections rather with the \mathcal{D} -modules itself. Recall that the left ideal defining the quotient $\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ is generated by the operators $\widehat{\square}_l$, $\widehat{E}_k - \beta_k z$ and $\widehat{E} - \beta_0 z$, where

$$\widehat{\square}_l := \prod_{i: l_i < 0} (z \cdot \partial_{\lambda_i})^{-l_i} - \prod_{i: l_i > 0} (z \cdot \partial_{\lambda_i})^{l_i},$$

$$\widehat{E} := z^2 \partial_z + \sum_{i=1}^t z \lambda_i \partial_{\lambda_i},$$

$$\widehat{E}_k := \sum_{i=1}^t b_{ki} z \lambda_i \partial_{\lambda_i}.$$

The presence of $z^{-2}(\widehat{E}_0 - \beta_0 z)$ in this ideal show that have the an isomorphism of $\mathbb{C}[z^\pm, \lambda_1, \dots, \lambda_n] \langle \partial_{\lambda_1}, \dots, \partial_{\lambda_n} \rangle$ -modules

$$\widehat{M} \simeq \mathbb{C}[z^\pm, \lambda_1, \dots, \lambda_n] \langle \partial_{\lambda_1}, \dots, \partial_{\lambda_n} \rangle / \mathbb{C}[z^\pm, \lambda_1, \dots, \lambda_n] \langle \partial_{\lambda_1}, \dots, \partial_{\lambda_n} \rangle \widehat{I} \quad (36)$$

where the left $\mathbb{C}[z^\pm, \lambda_1, \dots, \lambda_n] \langle \partial_{\lambda_1}, \dots, \partial_{\lambda_n} \rangle$ -ideal \widehat{I} is generated by $\widehat{\square}_{l \in \mathbb{L}}$ and $\widehat{E}_k - \beta_k$ for $k \in \{1, \dots, d\}$. The D_W -module corresponding to $\mathcal{H}^0 \left(i_1^+ \widehat{\mathcal{M}} \right)$ is given by $\widehat{M}/(z-1)\widehat{M}$. Using the isomorphism (36) one easily sees that

$$\widehat{M}/(z-1)\widehat{M} \simeq M_B^\beta,$$

which shows the claim. \square

Proposition 3.18. *The $\mathcal{O}_{\mathbb{C}_z \times W^\circ}$ -module ${}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ is locally-free of rank $\text{vol}(Q)$.*

Proof. Notice that it is sufficient to show that ${}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ is $\mathcal{O}_{\mathbb{C} \times W^\circ}$ -coherent. Namely, ${}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}/z \cdot {}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ is \mathcal{O}_{W° -locally free of rank $\text{vol}(Q)$ by Lemma 3.9. Moreover, the restriction of ${}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ to $\mathbb{C}_z^* \times W^\circ$ is a locally-free $\mathcal{O}_{\mathbb{C}_z^* \times W^\circ}$ -module by Lemma 3.13. Its restriction to $\{1\} \times W^\circ$ is isomorphic to the restriction of \mathcal{M}_B^β to W° by Lemma 3.17 which is locally free of rank $\text{vol}(Q)$. Now we use the fact that a coherent \mathcal{O} -module which has everywhere the same rank is locally-free.

It is actually sufficient to show the coherence of $\mathcal{N} := \text{For}_{z^2 \partial_z}({}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)})$, as this is the same as ${}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ when considered as an $\mathcal{O}_{\mathbb{C}_z \times W^\circ}$ -module. Let us denote by F_\bullet the natural filtration on $\mathcal{R}'_{\mathbb{C}_z \times W^\circ}$ defined by

$$F_k \mathcal{R}'_{\mathbb{C}_z \times W^\circ} := \left\{ P \in \mathcal{R}'_{\mathbb{C}_z \times W^\circ} \mid P = \sum_{|\alpha| \leq k} g_\alpha(z, \lambda) (z \partial_{\lambda_1})^{\alpha_1} \cdots (z \partial_{\lambda_t})^{\alpha_t} \right\}.$$

This filtration induces a filtration F_\bullet on \mathcal{N} which satisfies $F_k \mathcal{R}'_{\mathbb{C}_z \times W^\circ} \cdot F_l \mathcal{N} = F_{k+l} \mathcal{N}$. Obviously, for any k , $F_k \mathcal{N}$ is $\mathcal{O}_{\mathbb{C}_z \times W^\circ}$ -coherent, so that it suffices to show that the filtration F_\bullet becomes eventually stationary. Let $P = \sum_{|\alpha| \leq k} g_\alpha(z, \lambda) (z \partial_{\lambda_1})^{\alpha_1} \cdots (z \partial_{\lambda_t})^{\alpha_t}$ then its symbol is defined as

$$\sigma_k(P) := \sum_{|\alpha| = k} g_\alpha(z, \lambda) (\kappa_1)^{\alpha_1} \cdots (\kappa_t)^{\alpha_t} \in \mathcal{O}_{\mathbb{C}_z \times W^\circ}[\kappa_1, \dots, \kappa_t],$$

which is a function on $\mathbb{C}_z \times T^*W^\circ$ with fiber variables $\kappa_1, \dots, \kappa_t$. Let \mathcal{I} be the radical ideal of the ideal generated by the symbols of $\widehat{\square}_{l \in \mathbb{L}}$ and $\widehat{E}_k - z \cdot \beta_k$ for $k = 1, \dots, t$. Then the vanishing locus of \mathcal{I} is the $\mathcal{R}'_{\mathbb{C}_z \times W^\circ}$ -characteristic variety of \mathcal{N} . Notice that \mathcal{N} is $\mathcal{O}_{\mathbb{C}_z \times W^\circ}$ -coherent if and only if its $\mathcal{R}'_{\mathbb{C}_z \times W^\circ}$ -characteristic variety is a subset of $\mathbb{C}_z \times T_{W^\circ}^*W^\circ$. The proof of this fact is completely parallel to the \mathcal{D} -module case (see e.g. [Pha79, Proposition 10.3]).

To compute the $\mathcal{R}'_{\mathbb{C}_z \times W^\circ}$ -characteristic variety, notice that the symbols of $\widehat{\square}_{l \in \mathbb{L}}$ and $\widehat{E}_k - z \cdot \beta_k$ are independent of z . Thus it is enough to compute its restriction to $\{1\} \times W^\circ$. Now notice that the generators of the ideal corresponding to the GKZ-system \mathcal{M}_B^β have exactly the same symbols as the operators above. Thus it is enough to show that the restriction of the GKZ-system \mathcal{M}_B^β to W° is \mathcal{O}_{W° -coherent. But this follows from [Ado94, Lemma 3.2 and 3.3] and the definition of W° (see Definition 3.7 and Lemma 3.11). \square

Corollary 3.19. *The natural map ${}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} \rightarrow {}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}$ which is induced by the inclusion $\mathcal{R}_{\mathbb{C}_z \times W^*} \rightarrow \mathcal{D}_{\mathbb{C}_z \times W^*}$ is injective.*

Proof. Recall that $\mathcal{D}_{\mathbb{C}_z \times W^*} \otimes_{\mathcal{R}} {}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} \simeq \widehat{\mathcal{M}}_B^{(\beta_0, \beta)}|_{\mathbb{C}_z \times W^*}$ and $\mathcal{D}_{\mathbb{C}_z \times W^*} \simeq R[z^\pm]$. Thus the kernel of ${}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} \rightarrow \widehat{\mathcal{M}}_B^{(\beta_0, \beta)}|_{\mathbb{C}_z \times W^*}$ has z -torsion. On the open set $\mathbb{C}_z \times W^\circ \subset \mathbb{C}_z \times W^*$ the module ${}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)} = {}_0\widehat{\mathcal{M}}_B^{(\beta_0, \beta)}|_{\mathbb{C}_z \times W^\circ}$ is $\mathcal{O}_{\mathbb{C}_z \times W^\circ}$ -locally free. In particular it has no z -torsion, but this shows the claim. \square

The next step is to describe the image of ${}_0\widehat{\mathcal{M}}_B^{(0, \emptyset)}$ in \mathcal{G}^+ . In order to do this, consider once again the affine toric variety $X^{\text{aff}} = \text{Spec}(\mathbb{C}[\mathbb{N}B])$, which contains the torus $g_B(S) \cong S$ as an open subset (see formula (32)). Denote by D the complement of S in X^{aff} . We will consider X^{aff} as a log scheme in the sense of logarithmic geometry (see, e.g., [Gro11]). More precisely, we endow X^{aff} with divisorial log structure induced by D and W^* with the trivial log structure. We consider the relative log de Rham complex $\Omega_{X^{\text{aff}} \times W^*/W^*}^\bullet(\log D)$ ([Gro11, section 3.3]). We have isomorphisms $\Omega_{X^{\text{aff}} \times W^*/W^*}^k(\log D) \cong \mathcal{O}_{X^{\text{aff}} \times W^*} \otimes \bigwedge^k \mathbb{Z}^r$.

Proposition 3.20. *Let $\mathbb{N}B$ be a saturated semigroup. There exists the following $R_{\mathbb{C}_z \times W^\circ}$ -linear isomorphism*

$$H^0 \left(\Omega_{X^{\text{aff}} \times W^\circ / W^\circ}^{\bullet+s}(\log D)[z], zd - d_y F \wedge \right) \cong {}_0\widehat{M}_B^{(0,0)},$$

which maps ω_0 to 1.

Proof. We first define the $R_{\mathbb{C}_z \times W}$ -linear morphism

$$\begin{aligned} \psi : {}_0\widehat{M}_B^{(0,0)} &\longrightarrow H^0 \left(\Omega_{X^{\text{aff}} \times W^* / W^*}^{\bullet+s}(\log D)[z], zd - d_y F \wedge \right), \\ 1 &\mapsto \omega_0, \end{aligned}$$

which is well-defined by 3.4. Let

$$\omega = \sum_{\alpha, \gamma, \delta} c_{\alpha\gamma\delta} \lambda_1^{\gamma_1} \dots \lambda_t^{\gamma_t} z^\delta \underline{y}^{\alpha_1 \cdot b_1} \dots \underline{y}^{\alpha_t \cdot b_t} \omega_0$$

be a general element in $\Omega_{X^{\text{aff}} \times W^* / W^*}^s(\log D)[z]$ with $\alpha \in \mathbb{N}^t$, $\gamma \in \mathbb{Z}^t$ and $\delta \in \mathbb{N}$. Then

$$\sum_{\alpha, \gamma, \delta} c_{\alpha\gamma\delta} \lambda_1^{\gamma_1} \dots \lambda_t^{\gamma_t} z^\delta (z\partial_{\lambda_1})^{\alpha_1} \dots (z\partial_{\lambda_t})^{\alpha_t}$$

is a preimage, which shows that the map ψ is surjective. Notice that the restricted map

$${}^\circ\psi : {}_0\widehat{M}_B^{(0,0)} \longrightarrow H^0(\Omega_{X^{\text{aff}} \times W^\circ / W^\circ}^{\bullet+s}(\log D)[z], zd - d_y F \wedge)$$

is also surjective. Consider the following commutative diagram

$$\begin{array}{ccc} {}_0\widehat{M}_B^{(0,0)} & \xrightarrow{\cong} & H^0(\Omega_{S \times W^\circ / W^\circ}^{\bullet+s}[z^\pm], zd - d_y F \wedge) \\ \uparrow & & \uparrow \\ {}_0\widehat{M}_B^{(0,0)} & \xrightarrow{{}^\circ\psi} & H^0(\Omega_{X^{\text{aff}} \times W^\circ / W^\circ}^{\bullet+s}(\log D)[z], zd - d_y F \wedge) \end{array}$$

where the upper horizontal map is an isomorphism by Proposition 3.3 and Lemma 3.4, the left vertical map is injective by Corollary 3.19 and the right vertical map is induced by the morphism

$$\Omega_{X^{\text{aff}} \times W^\circ / W^\circ}^s(\log D)[z] \longrightarrow \Omega_{X^{\text{aff}} \times W^\circ / W^\circ}^s(*D)[z^\pm] = \Omega_{S \times W^\circ / W^\circ}^s[z^\pm].$$

But this shows that ${}^\circ\psi$ is also injective, which shows the claim. Notice that as a by-product, we also obtain that the morphism

$$H^0(\Omega_{X^{\text{aff}} \times W^\circ / W^\circ}^{\bullet+s}(\log D)[z], zd - d_y F \wedge) \longrightarrow H^0(\Omega_{S \times W^\circ / W^\circ}^{\bullet+s}[z^\pm], zd - d_y F \wedge)$$

is injective. □

4 Quantum cohomology of toric complete intersections

We recall in this section some more or less well known notations and results concerning so-called twisted Gromov-Witten invariants on the one hand, and basic constructions from toric geometry for smooth complete intersections in toric varieties on the other hand. The recent paper [MM11] can serve as a reference for both topics, however, we found it useful to collect here the material we need later in condensed form.

4.1 Twisted and reduced quantum \mathcal{D} -modules

A smooth complete intersection inside a smooth projective variety can be described as the zero locus of a generic section of a split vector bundle on that variety. Associated to such a bundle are the **twisted Gromov-Witten invariants**, which we describe first. They give rise to the twisted quantum product, and to the twisted quantum- \mathcal{D} -module. From this one can derive (basically by dividing by the kernel of the multiplication by the first Chern classes of the factors of the vector bundle) the **reduced quantum \mathcal{D} -module**, which corresponds, according to [MM11], to the ambient part of the quantum cohomology of the subvariety. We also discuss this reduced module here, and we define pairings (coming from the Poincaré pairing on the ambient variety) on both the twisted and the reduced quantum \mathcal{D} -module.

Let \mathcal{X} be a smooth projective n -dimensional variety. Let $\mathcal{L}_1, \dots, \mathcal{L}_c$ be line bundles on \mathcal{X} which are globally generated and define $\mathcal{E} := \bigoplus_{i=1}^s \mathcal{L}_s$. We want to repeat the construction of the so-called twisted quantum \mathcal{D} -module $\text{QDM}(\mathcal{X}, \mathcal{E})$ and the reduced quantum \mathcal{D} -module $\overline{\text{QDM}}(\mathcal{X}, \mathcal{E})$. A nice exposition can be found in [MM11, Chapter 2.5].

For $l \in \mathbb{N}$ and $d \in H_2(\mathcal{X}, \mathbb{Z})$ we denote by $\overline{\mathcal{M}}_{0,l,d}(\mathcal{X})$ the moduli space of stable maps of degree d from curves of genus 0 with l marked points to \mathcal{X} . Denote by $e_i : \overline{\mathcal{M}}_{0,l,d}(\mathcal{X}) \rightarrow \mathcal{X}$ the evaluation at the i marked point for $i \in \{1, \dots, l\}$ and denote by $\pi : \overline{\mathcal{M}}_{0,l+1,d}(\mathcal{X}) \rightarrow \overline{\mathcal{M}}_{0,l,d}(\mathcal{X})$ the map which forgets the last marked point. Let $\mathcal{E}_{0,l,d}$ be the locally free sheaf $R^0 \pi_* e_{l+1}^* \mathcal{E}$ and let $\mathcal{E}_{0,l,d}(l)$ be the kernel of the surjective morphism $\mathcal{E}_{0,l,d} \rightarrow e_{n+1}^* \mathcal{E}$ which evaluates a section at the l -marked point.

For $i \in \{1, \dots, l\}$ denote by \mathcal{N}_i the line bundle on $\overline{\mathcal{M}}_{0,l,d}(\mathcal{X})$ whose fiber at a point $(C, x_1, \dots, x_l, f : C \rightarrow \mathcal{X})$ is the cotangent space $T^*C_{x_i}$. Put $\phi_i := c_1(\mathcal{N}_i) \in H^2(\overline{\mathcal{M}}_{0,l,d}(\mathcal{X}))$.

Definition 4.1. Let $l \in \mathbb{N}$, $(m_1, \dots, m_l \in \mathbb{N}^l)$, $\gamma_1, \dots, \gamma_l \in H^{2*}(\mathcal{X})$ and $d \in H_2(\mathcal{X}, \mathbb{Z})$. The l -th twisted Gromov-Witten invariant with descendants is denoted by

$$\langle \tau_{m_1}(\gamma_1), \dots, \tau_{m_{l-1}}(\gamma_{l-1}), \widetilde{\tau_{m_l}(\gamma_l)} \rangle_{0,l,d} := \int_{[\overline{\mathcal{M}}_{0,l,d}(\mathcal{X})]^{vir}} c_{top}(\mathcal{E}_{0,l,d}(l)) \prod_{i=1}^l \phi_i^{m_i} e_i^* \gamma_i,$$

where $[\overline{\mathcal{M}}_{0,l,d}(\mathcal{X})]^{vir}$ is the virtual fundamental class of $\overline{\mathcal{M}}_{0,l,d}(\mathcal{X})$.

Let (T_0, T_1, \dots, T_h) be a homogeneous basis of $H^{2*}(\mathcal{X})$ such that $T_0 = 1$ and T_1, \dots, T_r is a basis of $H^2(\mathcal{X}, \mathbb{Z})$ modulo torsion which lies in the Kähler cone. Let T be the torus $H^2(\mathcal{X}, \mathbb{C})/2\pi i H^2(\mathcal{X}, \mathbb{Z})$. Then the basis T_1, \dots, T_r of $H^2(\mathcal{X}, \mathbb{Z})$ gives rise to coordinates $q = (q_1, \dots, q_r)$ on T .

Definition 4.2. Let $\gamma_1, \dots, \gamma_2 \in H^{2*}(\mathcal{X}, \mathbb{C})$ and $q \in T$. The twisted small quantum product is defined by

$$\gamma_1 \bullet_q^{tw} \gamma_2 := \sum_{a=1}^h \sum_{d \in H_2(\mathcal{X}, \mathbb{Z})} q^d \langle \gamma_1, \gamma_2, \tilde{T}_a \rangle_{0,3,d} T^a.$$

Let $\bar{T} = \mathbb{C}^r$ be a partial compactification of T with respect to the coordinates q_1, \dots, q_r . In the following we assume that there exists an open subset \bar{U} of \bar{T} such that the twisted quantum product is convergent. By [MM11, Proposition 2.14] the twisted quantum product is associative, commutative and has T_0 as a unit.

In analogy to the untwisted case one defines a trivial vector bundle F on $H^0(\mathcal{X}, \mathbb{C}) \times U \times \mathbb{C}$ with fiber $H^{2*}(\mathcal{X}, \mathbb{C})$ together with a flat meromorphic connection

$$\nabla_{\partial_{t_0}} := \partial_{t_0} + \frac{1}{z} T_0 \bullet_q^{tw}, \quad \nabla_{q_a \partial_{q_a}} := q_a \partial_{q_a} + \frac{1}{z} T_a \bullet_q^{tw}, \quad \nabla_{z \partial_z} := z \partial_z - \frac{1}{z} E \bullet_q^{tw} + \mu,$$

where μ is the diagonal morphism defined by $\mu(T_A) := \frac{1}{2}(\deg(T_A) - (\dim_{\mathbb{C}} \mathcal{X} - rk \mathcal{E}))T_A$ and $E := t_0 T_0 + c_1(\mathcal{T}_{\mathcal{X}} \otimes \mathcal{E}^{\vee})$ is the so-called Euler field.

Define a twisted pairing on $H^{2*}(\mathcal{X})$ by:

$$(\gamma_1, \gamma_2)^{tw} := \int_{\mathcal{X}} \gamma_1 \cup \gamma_2 \cup c_{top}(\mathcal{E}) \quad \text{for } \gamma_1, \gamma_2 \in H^{2*}(\mathcal{X}).$$

This pairing is degenerate with kernel equal to $\ker m_{c_{top}}$, where $m_{c_{top}}$ is defined by

$$m_{c_{top}} : H^{2*}(\mathcal{X}) \longrightarrow H^{2*}(\mathcal{X}),$$

$$\alpha \mapsto c_{top}(\mathcal{E}) \cup \alpha$$

and satisfies the Frobenius relation:

$$(\gamma_1 \bullet_q^{tw} \gamma_2, \gamma_3)^{tw} = (\gamma_1, \gamma_2 \bullet_q^{tw} \gamma_3)^{tw} \quad \text{for } \gamma_1, \gamma_2, \gamma_3 \in H^{2*}(\mathcal{X}).$$

Denote by \mathcal{F} the sheaf of global sections of F and define an involution ι by

$$\iota : H^0(\mathcal{X}) \times \mathbb{C}_z \times U \longrightarrow H^0(\mathcal{X}) \times \mathbb{C}_z \times U,$$

$$(t_0, z, q) \mapsto (t_0, -z, q).$$

We define a ∇ -flat sesquilinear pairing

$$S : \iota^*(\mathcal{F}) \times \mathcal{F} \longrightarrow \mathcal{O},$$

$$(s_1, s_2) \mapsto S(s_1, s_2)(t_0, z, q) = (s_1(t_0, -z, q), s_2(t_0, z, q))^{tw}.$$

We call $\overline{H^{2*}(\mathcal{X})} := H^{2*}(\mathcal{X}) / \ker m_{c_{top}}$ the reduced cohomology ring of $(\mathcal{X}, \mathcal{E})$. For $\gamma \in H^{2*}(\mathcal{X})$ denote by $\bar{\gamma}$ its class in $\overline{H^{2*}(\mathcal{X})}$. The pairing $(\cdot, \cdot)^{tw}$ gives rise to a pairing $(\cdot, \cdot)^{red}$ on $\overline{H^{2*}(\mathcal{X})}$ by

$$(\bar{\gamma}_1, \bar{\gamma}_2)^{red} := (\gamma_1, \gamma_2)^{tw} \quad \text{for } \gamma_1, \gamma_2 \in H^{2*}(\mathcal{X}).$$

Because the kernel of $(\cdot, \cdot)^{tw}$ is $\ker m_{c_{top}}$ this pairing is well-defined and non-degenerate. Denote by \overline{F} the trivial bundle on $H^0(\mathcal{X}) \times \mathbb{C}_z \times U$ with fiber $\overline{H^{2*}(\mathcal{X})}$. The pairing S induces a pairing \overline{S} on \overline{F} by

$$\overline{S}(\bar{s}_1, \bar{s}_2) := S(s_1, s_2),$$

which is non-degenerate.

Notice that $\overline{H^{2*}(\mathcal{X})}$ is naturally graded because $m_{c_{top}}$ is a graded morphism. Let $(\phi_0, \dots, \phi_{s'})$ be a homogeneous basis of $\overline{H^{2*}(\mathcal{X})}$ and denote by $(\phi^0, \dots, \phi^{s'})$ its dual basis w.r.t. $(\cdot, \cdot)^{red}$. The reduced Gromov-Witten invariants are defined by

$$\langle \bar{\gamma}_1, \dots, \bar{\gamma}_n \rangle_{0,l,d}^{red} := \langle \gamma_1, \dots, c_{top}(\mathcal{E}) \widetilde{\gamma}_n \rangle_{0,l,d}$$

and the reduced quantum product is

$$\bar{\gamma}_1 \bullet_q^{red} \bar{\gamma}_2 := \sum_{a=0}^{s'} \sum_{d \in H_2(\mathcal{X}, \mathbb{Z})} q^d \langle \bar{\gamma}_1, \bar{\gamma}_2, \phi_a \rangle_{0,3,d}^{red} \phi^a,$$

where the restriction is compatible with the multiplication, i.e.

$$\overline{\gamma_1 \bullet_q^{tw} \gamma_2} = \bar{\gamma}_1 \bullet_q^{red} \bar{\gamma}_2.$$

The bundle \overline{F} carries the following connection:

$$\overline{\nabla}_{\partial_{t_0}} := \partial_{t_0} + \frac{1}{z} \overline{T}_0 \bullet_q^{red}, \quad \overline{\nabla}_{q_a \partial_{q_a}} + \frac{1}{z} \overline{T}_a \bullet_q^{red}, \quad \overline{\nabla}_{z \partial_z} := z \partial_z - \frac{1}{z} \overline{E} \bullet_q^{red} + \overline{\mu},$$

where $\overline{\mu}$ is the diagonal morphism defined by $\overline{\mu}(\phi_a) := \frac{1}{2}(\deg(\phi_a) - (\dim_{\mathbb{C}} \mathcal{X} - rk \mathcal{E})) \phi_a$ and $\overline{E} := t_0 \overline{T}_0 + c_1(\overline{\mathcal{T}}_x \otimes \mathcal{E}^\vee)$. One can show that $\overline{\nabla}$ is flat and \overline{S} is $\overline{\nabla}$ -flat.

Definition 4.3. Consider the above situation of a smooth projective variety \mathcal{X} and globally generated line bundles $\mathcal{L}_1, \dots, \mathcal{L}_c$.

1. The triple (F, ∇, S) is called the twisted quantum \mathcal{D} -module $\text{QDM}(\mathcal{X}, \mathcal{E})$.
2. The triple $(\overline{F}, \overline{\nabla}, \overline{S})$ is called the reduced quantum \mathcal{D} -module $\overline{\text{QDM}}(\mathcal{X}, \mathcal{E})$.

4.2 Toric geometry of complete intersection subvarieties

In this subsection we consider the case where the variety \mathcal{X} from above is toric. It will be denoted by X_Σ , where Σ is the defining fan (see below). We recall some well-known results on the toric description of the total space of the bundle \mathcal{E} resp. its dual, on Picard groups, Kähler cones etc. All this is needed in section 6 below.

Let, as usual, N be a free abelian group of rank n for which we choose once and for all a basis which identifies it with \mathbb{Z}^n . Let Σ be a complete smooth fan in $N_{\mathbb{R}} := N \otimes \mathbb{R}$ and X_Σ the associated toric variety, which is compact and smooth. We recall the toric description of the Kähler resp. the nef cone of Σ . Let $\Sigma(1) = \{\mathbb{R}_{\geq 0}\underline{a}_1, \dots, \mathbb{R}_{\geq 0}\underline{a}_m\}$ be the rays of Σ , where $\underline{a}_i \in N \cong \mathbb{Z}^n$ are the primitive integral generators of the rays of Σ . Then we have an exact sequence

$$0 \longrightarrow \mathbb{L}_A \longrightarrow \mathbb{Z}^{\Sigma(1)} = \mathbb{Z}^m \longrightarrow N \longrightarrow 0, \quad (37)$$

where the morphism $\mathbb{Z}^m \rightarrow N$ is given by the matrix (henceforth called A) having the vectors $\underline{a}_1, \dots, \underline{a}_m$ as columns. \mathbb{L}_A is the module of relations between these vectors. We also consider the dual sequence

$$0 \longrightarrow M \longrightarrow (\mathbb{Z}^{\Sigma(1)})^\vee = \mathbb{Z}^m \longrightarrow \mathbb{L}_A^\vee \longrightarrow 0,$$

where $M := N^\vee$ is the dual lattice. It is well known that as X_Σ is smooth and compact, we have

$$H^2(X_\Sigma, \mathbb{Z}) \simeq \text{Pic}(X_\Sigma) \cong \mathbb{L}_A^\vee,$$

moreover, the group $(\mathbb{Z}^{\Sigma(1)})^\vee$ is the free abelian group generated by the torus invariant divisors on X_Σ . We denote these generators by D_1, \dots, D_m . Its images in \mathbb{L}_A^\vee (called \bar{D}_i) are thus the cohomology classes which are Poincaré dual to these divisors, and they generate the Picard group.

Any element in $(\mathbb{Z}^{\Sigma(1)})^\vee \otimes \mathbb{R}$ can be considered as a function on $N_{\mathbb{R}}$ (actually on the support of Σ , but this equals $N_{\mathbb{R}}$ by completeness), which is linear on each cone of Σ , these are called piecewise linear functions from now on and abbreviated by $\text{PL}(\Sigma)$. Inside $(\mathbb{Z}^{\Sigma(1)})^\vee \otimes \mathbb{R}$ we have the cone of convex functions, which are those functions $\psi \in \text{PL}(\Sigma)$ having the property that for any cone $\sigma \in \Sigma$ and for any $n \in N_{\mathbb{R}}$, we have $\psi(n) \leq \psi_\sigma(n)$, where ψ_σ is the extension to a linear function on all of $N_{\mathbb{R}}$ of the restriction $\psi|_\sigma$. The interior of the cone of convex functions are those which are strictly convex, that is, those such that the above inequality is strict on $\text{Int}(\sigma)$. Notice that any linear function on N is piecewise linear and this inclusion is precisely given by $M_{\mathbb{R}} \hookrightarrow (\mathbb{Z}^{\Sigma(1)})^\vee \otimes \mathbb{R}$. We define the nef cone \mathcal{K}_Σ of Σ to be the image under the projection $(\mathbb{Z}^{\Sigma(1)})^\vee \otimes \mathbb{R} \rightarrow \mathbb{L}_A^\vee \otimes \mathbb{R}$. Its interior is the Kähler cone \mathcal{K}_Σ° of Σ . We assume that \mathcal{K}_Σ° is non-empty, which amounts to say that X_Σ is projective. Let us recall the following description of the cone \mathcal{K}_Σ , the proof of this fact can be found, e.g., in [CK99, section 3.4.2].

Lemma 4.4. *For any cone $\sigma \in \Sigma$, put*

$$J_\sigma := \{i \in \{1, \dots, m\} \mid \mathbb{R}_{\geq 0}\underline{a}_i \notin \sigma\}$$

and define

$$\check{\sigma} := \sum_{i \in J_\sigma} \mathbb{R}_{\geq 0}\bar{D}_i \subset (\mathbb{L}_A^\vee)_{\mathbb{R}}.$$

We call $\check{\sigma}$ the anticone associated to σ . Then we have $\mathcal{K}_\Sigma = \bigcap_{\sigma \in \Sigma} \check{\sigma} \subset (\mathbb{L}_A^\vee)_{\mathbb{R}}$.

We proceed by considering the toric analogue of the situation from subsection 4.1. More precisely, let $\mathcal{L}_1, \dots, \mathcal{L}_c$ be line bundles on X_Σ . We suppose that the following two properties hold

Assumption 4.5.

1. For all $j = 1, \dots, c$, the line bundle \mathcal{L}_j is nef. Notice that according to [Ful93, Section 3.4], on a toric variety, \mathcal{L}_j is nef iff it is globally generated.
2. Let $-K_{X_\Sigma}$ be the anti-canonical divisor of X_Σ . Then we assume that $-K_{X_\Sigma} - \sum_{j=1}^c c_1(\mathcal{L}_j)$ is nef.

Put $\mathcal{E} := \bigoplus_{j=1}^c \mathcal{L}_j$ and consider the dual bundle $\mathcal{E}^\vee := \text{Hom}_{\mathcal{O}_{X_\Sigma}}(\mathcal{E}, \mathcal{O}_{X_\Sigma})$. We have the following fact.

Definition-Lemma 4.6. *The total space $\mathbb{V}(\mathcal{E}^\vee) := \text{Spec}_{\mathcal{O}_{X_\Sigma}}(\text{Sym}_{\mathcal{O}_{X_\Sigma}}(\mathcal{E}))$ of \mathcal{E}^\vee , is a smooth toric variety defined by a fan Σ' which is described in the following way. First we define the set of rays $\Sigma'(1)$: For this, we choose divisors $D_{m+j} = \sum_{i=1}^m d_{ji} D_i$ with $d_{ji} \geq 0$ and $\mathcal{O}(D_{m+j}) = \mathcal{L}_j$. This choice is possible due to Lemma 4.4 as all \mathcal{L}_j are nef. Write $\underline{d}_i := (d_{1i}, \dots, d_{ci}) \in \mathbb{Z}^c$ and put $\underline{a}'_i := (\underline{a}_i, \underline{d}_i) \in N' := N \times \mathbb{Z}^c \cong \mathbb{Z}^{n+c}$. Moreover, letting e_{n+1}, \dots, e_{n+c} be the last c standard generators of \mathbb{Z}^{n+c} , we put $\underline{a}'_{m+j} := e_{n+j}$. Then we let $\Sigma'(1) := \{\mathbb{R}_{\geq 0} \underline{a}'_1, \dots, \mathbb{R}_{\geq 0} \underline{a}'_{m+c}\}$ and we group, as before, the column vectors $\underline{a}'_1, \dots, \underline{a}'_{m+c}$ in a matrix $A' \in \text{Mat}((n+c) \times (m+c), \mathbb{Z})$. This means that*

$$A' = \left(\begin{array}{c|c} A & 0_{n,c} \\ \hline (d_{ji}) & \text{Id}_c \end{array} \right). \quad (38)$$

The fan Σ' is now defined as follows: For any set of vectors $\underline{b}_1, \dots, \underline{b}_r \in \mathbb{R}^k$ define $\langle \underline{b}_1, \dots, \underline{b}_r \rangle := \sum_{j=1}^r \mathbb{R}_{\geq 0} \underline{b}_j$. Then we put

$$\Sigma' := \{ \langle \underline{a}'_{i_1}, \dots, \underline{a}'_{i_k}, \underline{e}_{j_1}, \dots, \underline{e}_{j_t} \rangle \subset N'_\mathbb{R} \mid \langle \underline{a}_{i_1}, \dots, \underline{a}_{i_k} \rangle \in \Sigma(k), \{j_1, \dots, j_t\} \subset \{n+1, \dots, n+c\} \}.$$

In other words, considering the canonical projection $\pi : N'_\mathbb{R} \rightarrow N_\mathbb{R}$ which forgets the last c components, we have that $\sigma' \in \Sigma'$ iff $\pi(\sigma') \in \Sigma$.

In the following proposition, we list some rather obvious properties of the cohomology (resp. its toric description) of the space $\mathbb{V}(\mathcal{E}^\vee)$.

Proposition 4.7. *Let X_Σ , $\mathcal{L}_1, \dots, \mathcal{L}_c$ and the sum \mathcal{E} resp. its dual \mathcal{E}^\vee be as above.*

1. *The projection map $p : \mathbb{V}(\mathcal{E}^\vee) \rightarrow X_\Sigma$ induces an isomorphism $p^* : H^*(X_\Sigma, \mathbb{Z}) \cong H^*(\mathbb{V}(\mathcal{E}^\vee), \mathbb{Z})$.*
2. *Consider the analogue of sequence (37) for the matrix A' , that is, the sequence*

$$0 \longrightarrow \mathbb{L}_{A'} \longrightarrow \mathbb{Z}^{\Sigma'(1)} = \mathbb{Z}^{m+c} \longrightarrow N' \longrightarrow 0, \quad (39)$$

then we have an isomorphism

$$\begin{aligned} \mathbb{L}_A &\longrightarrow \mathbb{L}_{A'} \\ \underline{l} = (l_1, \dots, l_m) &\longmapsto \underline{l}' := (l_1, \dots, l_m, l_{m+1}, \dots, l_{m+c}), \end{aligned} \quad (40)$$

where $l_{m+j} := -\sum_{i=1}^m l_i d_{ji} = -\langle c_1(\mathcal{L}_j), \underline{l} \rangle$ for all $j = 1, \dots, c$, and where $\langle -, - \rangle$ is the non-degenerate intersection product between $\mathbb{L} \cong H_2(X_\Sigma, \mathbb{Z})$ and $\text{Pic}(X_\Sigma)$. Notice that in the definition of this isomorphism we consider \mathbb{L}_A resp. $\mathbb{L}_{A'}$ as embedded into \mathbb{Z}^m resp. \mathbb{Z}^{m+c} .

3. *The scalar extension $H^2(X_\Sigma, \mathbb{R}) \xrightarrow{\cong} H^2(\mathbb{V}(\mathcal{E}^\vee), \mathbb{R})$ of the isomorphism p^* from above identifies the Kähler cones (resp. the nef cones) $\mathcal{K}_{X_\Sigma}^\circ$ and $\mathcal{K}_{\mathbb{V}(\mathcal{E}^\vee)}^\circ$ (resp. \mathcal{K}_{X_Σ} and $\mathcal{K}_{\mathbb{V}(\mathcal{E}^\vee)}$).*
4. *The manifold $\mathbb{V}(\mathcal{E}^\vee)$ is nef. Moreover, if $s \in \Gamma(X_\Sigma, \mathcal{E})$ is generic, and $Y := s^{-1}(0)$ is the zero locus of this section, then also Y is smooth and also nef.*

Proof. The first point follows from the fact that $\mathbb{V}(\mathcal{E}^\vee)$ and X_Σ are homotopy equivalent. The second point follows from a direct calculation. For the third point notice that the isomorphism p^* restricted to $H^2(X_\Sigma)$ is given by

$$\begin{aligned} p^* : H^2(X_\Sigma) &\simeq \bigoplus_{i=1}^m \mathbb{Z} D_i / \left(\sum_{i=1}^m a_{ki} D_i \right)_{k=1, \dots, n} \longrightarrow \bigoplus_{i=1}^{m+c} \mathbb{Z} D'_i / \left(\sum_{i=1}^{m+c} a'_{ki} D'_i \right)_{k=1, \dots, n+c} \simeq H^2(\mathbb{V}(\mathcal{E}^\vee)), \\ &\sum_{i=1}^m d_i D_i \longmapsto \sum_{i=1}^m d_i D'_i. \end{aligned}$$

We first prove $p^*(\mathcal{K}_{X_\Sigma}) \subset \mathcal{K}_{\mathbb{V}(\mathcal{E}^\vee)}$. Let $D = \sum_{i=1}^m d_i D_i$ be a divisor in X_Σ with $\overline{D} \in \mathcal{K}_{X_\Sigma}$. Then ψ_D^Σ is given on a maximal cone $\sigma \in \Sigma(n)$ by $u_\sigma^\Sigma \in M \simeq \mathbb{Z}^n$ which is defined by $\langle u_\sigma^\Sigma, \underline{a}_i \rangle = -d_i$ for $\underline{a}_i \in \sigma$. The PL-function ψ_D^Σ is convex if and only if for all $\sigma \in \Sigma(n)$ the following inequalities hold $\langle u_\sigma^\Sigma, \underline{a}_i \rangle \geq -d_i$

for all $i \in \{1, \dots, m\}$. Now consider the corresponding PL-function $\psi_{p^*(D)}^{\Sigma'}$ for $p^*(D)$. Let $\sigma' \in \Sigma'(n+c)$ be a maximal cone in Σ' with $\sigma' = \langle \underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \underline{e}_{n+c} \rangle$, where $\{i_1, \dots, i_n\} \subset \{1, \dots, m\}$. Then $u_{\sigma'}^{\Sigma'} \in M' \simeq \mathbb{Z}^{n+c}$ is defined by

$$\langle u_{\sigma'}^{\Sigma'}, \underline{a}'_i \rangle = -d_i \quad \text{for } i \in \{i_1, \dots, i_n\}$$

and

$$\langle u_{\sigma'}^{\Sigma'}, \underline{e}_i \rangle = 0 \quad \text{for } i \in \{n+1, \dots, n+c\}. \quad (41)$$

But because of equation (41) we have

$$\langle u_{\sigma'}^{\Sigma'}, \underline{a}'_i \rangle = \langle u_{\sigma}^{\Sigma}, \underline{a}_i \rangle \geq -d_i \quad \text{for } i \in \{1, \dots, m\},$$

which shows that $\psi_{p^*(D)}^{\Sigma'}$ is convex, i.e. $p^*(\overline{D}) \in \mathcal{K}_{\mathbb{V}(\mathcal{E}^\vee)}$. Now assume $\overline{D'} \in \mathcal{K}_{\mathbb{V}(\mathcal{E}^\vee)}$. Because p^* is an isomorphism, we can assume that D' has a presentation $\sum_{i=1}^{m+c} d'_i D'_i$ in which $d'_{m+j} = 0$ for $j \in \{1, \dots, c\}$, i.e. $\overline{D'} = p^*(\overline{D})$ with $D = \sum_{i=1}^m d'_i D_i$. Let $\sigma \in \Sigma(n)$ and $\sigma' \in \Sigma(n+c)$ maximal cones with $\pi(\sigma') = \sigma$. Because of the presentation of D' we have $\langle u_{\sigma'}^{\Sigma'}, \underline{e}_i \rangle = 0$ for $i \in \{n+1, \dots, n+c\}$. Therefore we have

$$\langle u_{\sigma}^{\Sigma}, \underline{a}_i \rangle = \langle u_{\sigma'}^{\Sigma'}, \underline{a}'_i \rangle \geq -d_i,$$

which shows that $\psi_{\overline{D}}^{\Sigma}$ is convex, i.e. $\overline{D} \in \mathcal{K}_{X_\Sigma}$. The statement for the open parts follows from the fact that p^* is a homeomorphism.

For the fourth point recall that $\mathbb{V}(\mathcal{E}^\vee)$ is nef, i.e. has a nef anticanonical divisor, if the class of the divisor

$$-K_{\mathbb{V}(\mathcal{E}^\vee)} = \sum_{i=1}^m D'_i + \sum_{j=1}^c D'_{m+j}$$

lies in $\mathcal{K}_{\mathbb{V}(\mathcal{E}^\vee)}$. Because of 3. it is enough to show that $(p^*)^{-1}(-K_{\mathbb{V}(\mathcal{E}^\vee)})$ lies in \mathcal{K}_{X_Σ} . But we have

$$(p^*)^{-1}(-\overline{K}_{\mathbb{V}(\mathcal{E}^\vee)}) = \sum_{i=1}^m \overline{D}_i - \sum_{j=1}^c \sum_{i=1}^m d_{ji} \overline{D}_i = -\overline{K}_{X_\Sigma} - \sum_{j=1}^c c_1(\mathcal{L}_j)$$

and the term on the right hand side lies in \mathcal{K}_{X_Σ} by Assumption 4.5 2. Let $s \in \Gamma(X_\Sigma, \mathcal{E})$ be a generic section, then one can show that $Y = s^{-1}(0)$ is smooth by repeatedly applying Bertini's theorem. The nefness of Y is obtained by repeatedly applying the adjunction formula and Assumption 4.5 2. \square

We finish this section by the following remark, which will not be explicitly used in the sequel, but which helps to understand the geometry of the torus embedding considered in the beginning of section 2. More precisely, let $S := \text{Spec } \mathbb{C}[\mathbb{Z}^{n+c}]$ and denote again by $g : S \rightarrow \mathbb{P}^{m+c}$ the map defined by $(y_1, \dots, y_{m+c}) \mapsto (1 : \underline{y}^{\underline{a}'_1} : \dots : \underline{y}^{\underline{a}'_{m+c}})$. In section 2 we considered the factorization $g : S \xrightarrow{j} X \xrightarrow{i} \mathbb{P}^{m+c}$ (with $X := \overline{\text{Im}}(g)$) where j is an open embedding and i is a closed embedding. However, we will also need to consider some other factorization, namely, we write $g = g^{(2)} \circ g^{(1)}$, where $g^{(1)} : S \rightarrow \mathbb{C}^m \times (\mathbb{C}^*)^c$ sends \underline{y} to $(y^{\underline{a}'_i})_{i=1, \dots, m+c}$ and $g^{(2)}$ is the composition of the two open embeddings $\mathbb{C}^m \times (\mathbb{C}^*)^c \hookrightarrow \mathbb{C}^{m+c}$ and $\mathbb{C}^{m+c} \hookrightarrow \mathbb{P}^{m+c}$. Now we have the following fact.

Proposition 4.8. *The morphism $g^{(1)}$ is a closed embedding. Hence, we have*

$$X \setminus \text{Im}(g) \subset \{\mu_0 \cdot \mu_{m+1} \cdots \mu_{m+c} = 0\},$$

where we use $(\mu_0 : \mu_1 : \dots : \mu_{m+c})$ as homogeneous coordinates on \mathbb{P}^{m+c} and μ_1, \dots, μ_m as coordinates on \mathbb{C}^{m+c} (resp. on $(\mathbb{C}^*)^{m+c}$, $\mathbb{C}^m \times (\mathbb{C}^*)^c$ etc).

Proof. It suffices obviously to show the first statement. We will use a method similar to the proof of [RS10, Proposition 2.1]. First notice that the embedding $\alpha : S \hookrightarrow (\mathbb{C}^*)^{m+c}$ sending \underline{y} to $(\underline{y}^{\underline{a}'_i})_{i=1, \dots, m+c}$ is obviously closed, so that it suffices to show that $\overline{\text{im}}(g^{(1)}) \cap (\mathbb{C}^m \setminus (\mathbb{C}^*)^m) \times (\mathbb{C}^*)^c = \emptyset$. Recall that $\text{im}(g^{(1)})$ is the closed subvariety of $\mathbb{C}^m \times (\mathbb{C}^*)^c$ defined by the binomial equations

$$\prod_{i:l'_i > 0} \mu_i^{l'_i} - \prod_{i:l'_i < 0} \mu_i^{-l'_i}$$

for any $l' \in \mathbb{L}_{A'}$ (these equations form the toric ideal of A'). It was shown in loc.cit. that due to the compactness of X_Σ , there is some \underline{l} lying in $\mathbb{L}_A \cap \mathbb{Z}_{>0}^m$. Hence, the image l' of \underline{l} under the isomorphism (40) lies in $\mathbb{Z}_{>0}^m \times \mathbb{Z}_{<0}^c$, as the coefficients d_{ji} appearing in formula (40) are non-negative (see Definition 4.6) and moreover, for fixed j , not all d_{ji} can be zero. It follows that the toric ideal of A' contains an equation

$$\prod_{i=1}^m \mu_i^{l'_i} - \prod_{i=m+1}^{m+c} \mu_i^{-l'_i}, \quad (42)$$

where none of the exponents is zero. Now suppose that there is a point $x = (x_1, \dots, x_m, x_{m+1}, \dots, x_{m+c}) \in \overline{\text{Im}(g^{(1)})} \cap (\mathbb{C}^m \setminus (\mathbb{C}^*)^m) \times (\mathbb{C}^*)^c$, that is, we have $x_i = 0$ for some $i \in \{1, \dots, m\}$, then as equation (42) vanishes on x , we must have some $j \in \{1, \dots, c\}$ with $x_{m+j} = 0$, which contradicts the assumption that $x \in (\mathbb{C}^m \setminus (\mathbb{C}^*)^m) \times (\mathbb{C}^*)^c$. Hence the intersection $\overline{\text{Im}(g^{(1)})} \cap (\mathbb{C}^m \setminus (\mathbb{C}^*)^m) \times (\mathbb{C}^*)^c$ is indeed empty from which it follows that $g^{(1)} : S \hookrightarrow \mathbb{C}^m \times (\mathbb{C}^*)^c$ is a closed embedding. \square

Remark: The GKZ-systems (see Definition 2.8) associated to the matrix A' is not necessary regular, as the vectors $\underline{a}'_1, \dots, \underline{a}'_{m+c}$ do not necessarily lie on an affine hyperplane in \mathbb{Z}^{m+c} (see [Hot98] for this regularity criterion). The situation is similar to that considered in our earlier paper [RS10], and for the same reasons as in loc.cit., we will work with the extended matrix $A'' \in \text{Mat}((1+n+c) \times (1+m+c), \mathbb{Z})$ with columns $\underline{a}''_0, \underline{a}''_1, \dots, \underline{a}''_{m+c}$, where $\underline{a}''_i := (1, \underline{a}'_i)$ and $\underline{a}''_0 := (1, \underline{0}, \underline{0})$. In particular we have $\underline{a}''_{m+j} = (1, \underline{e}_{n+j}) \in \mathbb{Z}^{n+c+1}$ for $j = 1, \dots, c$ where \underline{e}_{n+j} is the $n+j$ -th standard vector in \mathbb{C}^{n+c} . We write $\mathbb{L}_{A''}$ for the module of relations between the columns of A'' , obviously we have an isomorphism $\mathbb{L}_{A'} \rightarrow \mathbb{L}_{A''}$ sending $\underline{l} = (l_1, \dots, l_{m+c})$ to $(-\sum_{i=1}^{m+c} l_i, \underline{l})$. As a matter of notation, we will often write the parameter of the GKZ-systems defined by the matrix A'' , which are vectors in \mathbb{C}^{1+m+c} by definition, as $(\alpha, \underline{\beta}, \underline{\gamma}) \in \mathbb{C}^{1+m+c}$, where $\alpha \in \mathbb{C}$, $\underline{\beta} \in \mathbb{C}^m$ and $\underline{\gamma} \in \mathbb{C}^c$.

5 Euler-Koszul homology and duality of GKZ-systems

In this section, we show a duality result for the GKZ-systems associated to the toric situation just described. We will explain how to calculate the holonomic dual of the system $\mathcal{M}_{A''}^\beta$ for some specific β , this is used to get a more precise description of the various \mathcal{D} -module considered in sections 2 and 3. The methods used here are close to our earlier paper [RS10], but due to non-compactness of the toric varieties involved, the proofs are more complicated.

Proposition 5.1. *Let X_Σ , $\mathcal{L}_1, \dots, \mathcal{L}_c$ as in section 4. Let A' be the matrix from in Definition 4.6 (i.e. with columns the primitive integral generator of the fan of $\mathbb{V}(\mathcal{E}^\vee)$) and A'' its extension considered at the end of section 4. Then we have*

1. The semigroup $\mathbb{N}A''$ is normal and the map

$$\begin{aligned} \Psi : \mathbb{N}A'' &\longrightarrow (\mathbb{N}A'')^\circ, \\ \underline{m} &\longmapsto \underline{m} + \underline{a}''_0 + \underline{a}''_{m+1} + \dots + \underline{a}''_{m+c} \end{aligned}$$

is an isomorphism of semigroups. Hence the semigroup ring $\mathbb{C}[\mathbb{N}A'']$ is normal, Cohen-Macaulay and Gorenstein.

2. The semigroup $\mathbb{N}A'$ is also normal and Cohen-Macaulay. We have that

$$(\mathbb{N}A')^\circ = \underline{a}'_{m+1} + \dots + \underline{a}'_{m+c} + \mathbb{N}A'$$

that is, any $\beta \in \sum_{j=1}^c \underline{a}'_{m+j} + \mathbb{N}A' \subset \mathbb{Z}^{n+c}$ lies in $(\mathbb{N}A')^\circ$.

Before entering into the proof, we need some notations and preliminary results. For any finite set $\underline{b}_1, \dots, \underline{b}_r \in \mathbb{R}^k$, put $C(\{\underline{b}_1, \dots, \underline{b}_r\}) := \sum_{j=1}^r \mathbb{R}_{\geq 0} \underline{b}_j$ and let $\text{Conv}(\{\underline{b}_1, \dots, \underline{b}_r\}) := \{\sum_{j=1}^r \lambda_j \underline{b}_j \mid \lambda_j \geq 0, \sum_{j=1}^r \lambda_j = 1\}$ be the convex hull of $\underline{b}_1, \dots, \underline{b}_r$. Notice that in particular we have $\text{Conv}(\{\underline{0}, \underline{b}_1, \dots, \underline{b}_r\}) := \{\sum_{j=1}^r \lambda_j \underline{b}_j \mid \lambda_j \geq 0, \sum_{j=1}^r \lambda_j \leq 1\}$ As a piece of notation, given a matrix $B \in \text{M}(k \times r, \mathbb{R})$ with columns $\underline{b}_1, \dots, \underline{b}_r$, write $C(B) := C(\{\underline{b}_1, \dots, \underline{b}_r\})$ and $\text{Conv}(B) := \text{Conv}(\{\underline{b}_1, \dots, \underline{b}_r\})$. Recall that for any cone

$\langle \underline{a}'_{i_1}, \dots, \underline{a}'_{i_{n+c}} \rangle \in \Sigma'(n+c)$, we use the convention that $\underline{a}'_{i_k} = (\underline{a}_{i_k}, \underline{d}_{i_k})$ for $k = 1, \dots, n$ and $i_{n+j} = m+j$ for $j = 1, \dots, c$, that is, $\underline{a}'_{i_{n+j}} = \underline{e}_{n+j}$ for $j = 1, \dots, c$. In particular, we have that $\underline{a}''_{i_k} = (1, \underline{a}_{i_k}, \underline{d}_{i_k}) \in \mathbb{Z}^{n+c+1}$ for $k = 1, \dots, n$ and $\underline{a}''_{i_{n+l}} = (1, \underline{e}_{m+l}) \in \mathbb{Z}^{n+c+1}$.

Let i_1, \dots, i_k be any subset of $\{0, \dots, m+c\}$, then by putting $\underline{a}'_0 := \underline{0} \in N'$ we clearly have

$$C(\{\underline{a}''_{i_1}, \dots, \underline{a}''_{i_k}\}) \stackrel{!}{=} \bigcup_{\substack{\underline{x}' \in \text{Conv}(\underline{a}'_{i_1}, \dots, \underline{a}'_{i_k}) \\ \lambda \in \mathbb{R}_{\geq 0}}} \lambda \cdot (1, \underline{x}'). \quad (43)$$

The following result is needed later in the toric characterization of the nef condition for the space $\mathbb{V}(\mathcal{E}^\vee)$. It applies in a slightly more general situation as the one considered so far, namely, let X_Σ be toric as before and let $\mathcal{L}_1, \dots, \mathcal{L}_c$ be nef line bundles. However, we do not suppose that $-K_{X_\Sigma} - \sum_{j=1}^c \mathcal{L}_j$ is nef. An example where the following lemma applies and which falls out of the scope of the setting of Proposition 5.1 is $X_\Sigma = \mathbb{P}^1$, $c = 1$ and $\mathcal{L}_1 := \mathcal{O}_{\mathbb{P}^1}(k)$ for $k > 2$.

Lemma 5.2. *Let as before X_Σ toric and $\mathcal{L}_1, \dots, \mathcal{L}_c$ nef (but $-K_{X_\Sigma} - \sum_{j=1}^c \mathcal{L}_j$ can be arbitrary). Consider the fan Σ' of the space $\mathbb{V}(\mathcal{E}^\vee)$, where $\mathcal{E} = \bigoplus_{j=1}^c \mathcal{L}_j$. Then we have*

$$\text{Conv}(0, \underline{a}'_1, \dots, \underline{a}'_m, \underline{a}'_{m+1}, \dots, \underline{a}'_{n+c}) = \text{Conv}(0, \underline{a}'_1, \dots, \underline{a}'_m, \underline{e}_{n+1}, \dots, \underline{e}_{n+c}) \stackrel{!}{\subset} \text{Supp}(\Sigma').$$

Proof. First notice that due to the special shape of the matrix A' , we have that for any point $\underline{x}' \in \text{Supp}(\Sigma')$, any $j = 1, \dots, c$ and any $\varepsilon \geq 0$, the point $\underline{x}' + \varepsilon \underline{e}_{n+j}$ is also an element of $\text{Supp}(\Sigma')$.

Consider a point $\underline{y}' = \sum_{i=1}^{m+c} \tilde{\lambda}_i \underline{a}'_i$ with $\tilde{\lambda}_i \in [0, 1]$ and $\sum_{i=1}^{m+c} \tilde{\lambda}_i \leq 1$, that is,

$$\underline{y}' \in \text{Conv}(0, \underline{a}'_1, \dots, \underline{a}'_m, \underline{e}_{n+1}, \dots, \underline{e}_{n+c}),$$

and suppose that $\underline{y}' \notin \text{Supp}(\Sigma')$. This implies in particular that there is some $i \in \{1, \dots, m\}$ with $\tilde{\lambda}_i \neq 0$, hence, $\sum_{i=1}^m \tilde{\lambda}_i \neq 0$. Notice that for all $\underline{p}' \in \text{Supp}(\Sigma')$ we have $\delta \cdot \underline{p}' \in \text{Supp}(\Sigma')$ for any $\delta \in \mathbb{R}_{\geq 0}$. Hence the point $\underline{x}' := \lambda \cdot \underline{y}'$ where $\lambda := \left(\sum_{i=1}^m \tilde{\lambda}_i\right)^{-1}$ does not lie in $\text{Supp}(\Sigma')$.

Let \underline{x} be the image of \underline{x}' under the canonical projection $N'_\mathbb{R} \rightarrow N_\mathbb{R}$ which forgets the last c coordinates. Putting $\lambda_i := \lambda \cdot \tilde{\lambda}_i$, it follows that $\underline{x} = \sum_{i=1}^m \lambda_i \underline{a}_i$ and that $\sum_{i=1}^m \lambda_i = 1$. Due to the completeness of Σ , we have $\underline{x} \in \text{Supp}(\Sigma)$ and hence there is some maximal cone $\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle$ in Σ containing the point \underline{x} . We have that $\underline{x}' \notin \langle \underline{a}'_1, \dots, \underline{a}'_m, \underline{e}_{n+1}, \dots, \underline{e}_{n+c} \rangle$ for otherwise \underline{x}' would lie in $\text{Supp}(\Sigma')$. For any $j = 1, \dots, c$, consider the piecewise linear function $\psi_{\mathcal{L}_j}$ on $\text{Supp}(\Sigma)$ (these were called $\text{PL}(\Sigma)$ in the discussion in subsection 4.2) associated to the divisor $\sum_{i=1}^m d_{ji} D_i$ (see Definition 4.6 and the proof of Proposition 4.7). By the assumption, the line bundles \mathcal{L}_j are nef and hence the functions $\psi_{\mathcal{L}_j}$ are convex on $\text{Supp}(\Sigma)$. This implies in particular that

$$\psi_{\mathcal{L}_j}(\underline{x}) \geq \sum_{i=1}^m \lambda_i \psi_{\mathcal{L}_j}(\underline{a}_i) = - \sum_{i=1}^m \lambda_i d_{ji}.$$

From $\underline{x} \in \langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle$ we deduce that there are $\mu_1, \dots, \mu_n \in \mathbb{R}_{\geq 0}$ with $\underline{x} = \sum_{k=1}^n \mu_k \underline{a}_{i_k}$ and thus $\psi_{\mathcal{L}_j}(\underline{x}) = - \sum_{k=1}^n \mu_k d_{ji_k}$. We hence arrive at the inequality

$$\sum_{k=1}^n \mu_k d_{ji_k} \leq \sum_{i=1}^m \lambda_i d_{ji}. \quad (44)$$

On the other hand, we have that

$$\sum_{k=1}^n \mu_k \underline{a}'_{i_k} = \left(\underline{x}, \left(\sum_{k=1}^n \mu_k d_{ji_k} \right)_{j=1, \dots, c} \right) \in \text{Supp}(\Sigma')$$

so that we conclude by the inequality (44) and the first remark of this proof that

$$\left(\underline{x}, \left(\sum_{i=1}^m \lambda_i d_{ji} \right)_{j=1, \dots, c} \right) = \underline{x}' \in \text{Supp}(\Sigma')$$

contradicting the assumption. \square

Now we come back to our original situation, where both the individual bundles \mathcal{L}_j and the divisor $-K_{\mathbb{V}(\mathcal{E}^\vee)} = -K_{X_\Sigma} - \sum_{j=1}^c c_1(\mathcal{L}_j)$ are nef (see Proposition 4.7, 4.). We introduce the following set

$$W(\Sigma') := \bigcup_{\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle \in \Sigma(n)} \text{Conv}(\underline{0}, \underline{a}'_{i_1}, \dots, \underline{a}'_{i_n}, \underline{e}_{n+1}, \dots, \underline{e}_{n+c}).$$

Lemma 5.3. *Let X_Σ and $\mathcal{L}_1, \dots, \mathcal{L}_c$ be as in section 4. Then the set $W(\Sigma')$ is convex and we have*

$$W(\Sigma') = \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_{m+c}). \quad (45)$$

Proof. First notice that $W(\Sigma') \subset \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_{m+c})$ always holds true, because the set of the right hand side is convex by definition and contains $\underline{0}, \underline{a}'_{i_1}, \dots, \underline{a}'_{i_n}, \underline{e}_{n+1}, \dots, \underline{e}_{n+c}$ for any $\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle \in \Sigma(n)$, hence it must contain the convex hull of these vectors, i.e., the set $\text{Conv}(\underline{0}, \underline{a}'_{i_1}, \dots, \underline{a}'_{i_n}, \underline{e}_{n+1}, \dots, \underline{e}_{n+c})$. On the other hand, once we know that $W(\Sigma')$ is convex, it must contain the convex hull of all its points, in particular, the convex hull of $\underline{0}, \underline{a}'_1, \dots, \underline{a}'_{m+c}$, i.e., the set $\text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_{m+c})$.

We are thus left to show the convexity of $W(\Sigma')$. Notice that this property would follow from (and in fact, would be equivalent to) the nef property of $\mathbb{V}(\mathcal{E}^\vee)$ if $\mathbb{V}(\mathcal{E}^\vee)$ were complete (see [Wis02, page 268]). However, putting $X_\Sigma := \mathbb{F}_3$ (the third Hirzebruch surface) and $\mathcal{L}_1 := -K_{\mathbb{F}_3}$ we obtain a quasi-projective smooth Calabi-Yau (in particular, nef) variety $\mathbb{V}(K_{\mathbb{F}_3})$ with defining fan Σ' such that $W(\Sigma')$ is not convex (and, as one easily checks, the semi-group NA'' is not normal). Notice that for this example neither the condition nor the conclusion of Lemma 5.2 is satisfied (namely, $-K_{\mathbb{F}_3}$ is not nef and we have $\text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_4, \underline{e}_3) \not\subset \text{Supp}(\Sigma')$). Similarly, we can consider $X_\Sigma = \mathbb{P}^1$, $c = 1$ and $\mathcal{L}_1 = \mathcal{O}_{\mathbb{P}^1}(k)$ for $k < 0$, then $-K_{\mathbb{P}^1} - \mathcal{L}_1 = \mathcal{O}_{\mathbb{P}^1}(k+2)$ is ample (hence nef), but \mathcal{L}_1 itself is not nef, we have that $\text{Conv}(\underline{0}, \underline{a}'_1, \underline{a}'_2, \underline{e}_2) \not\subset \text{Supp}(\Sigma')$ and moreover, $W(\Sigma')$ is not convex.

We will show the convexity of $W(\Sigma')$ by contradiction. Suppose that $W(\Sigma')$ is not convex, then there are two points $\underline{a}', \underline{b}' \in W(\Sigma')$ and some real number $t \in (0, 1)$ such that $\underline{x}' := t \cdot \underline{a}' + (1-t) \cdot \underline{b}' \notin W(\Sigma')$. $\underline{a}', \underline{b}'$ being elements in $W(\Sigma')$ means that there are two cones $\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle, \langle \underline{a}'_{i'_1}, \dots, \underline{a}'_{i'_n} \rangle \in \Sigma(n)$, such that

$$\underline{a}' = \sum_{k=1}^n \lambda_k \underline{a}'_{i_k} + \sum_{j=1}^c \lambda_{n+j} \underline{e}_{n+j} \quad \underline{b}' = \sum_{k=1}^n \mu_k \underline{a}'_{i'_k} + \sum_{j=1}^c \mu_{n+j} \underline{e}_{n+j}$$

with $\lambda_k, \mu_k \in \mathbb{R}_{\geq 0}$, $\sum_{k=1}^{n+c} \lambda_k \leq 1$ and $\sum_{k=1}^{n+c} \mu_k \leq 1$. From this description it is clear that

$$\underline{x}' \in \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_m, \underline{e}_{n+1}, \dots, \underline{e}_{n+c})$$

and hence by Lemma 5.2 (which applies as $\mathcal{L}_1, \dots, \mathcal{L}_c$ are supposed to be nef) we have $\underline{x}' \in \text{Supp}(\Sigma')$. We claim that the piecewise linear function on $\psi_{-K_{\mathbb{V}(\mathcal{E}^\vee)}}$ on $\text{Supp}(\Sigma')$ defined by $\psi_{-K_{\mathbb{V}(\mathcal{E}^\vee)}}(\underline{a}'_i) := -1$ for $i = 1, \dots, m+c$ is then not convex (see again the discussion in subsection 4.2), and thus the variety $\mathbb{V}(\mathcal{E}^\vee)$ is not nef, contradiction our assumption. Convexity of $\psi_{-K_{\mathbb{V}(\mathcal{E}^\vee)}}$ would imply that

$$\psi_{-K_{\mathbb{V}(\mathcal{E}^\vee)}}(\underline{x}') \geq t \cdot \psi_{-K_{\mathbb{V}(\mathcal{E}^\vee)}}(\underline{a}') + (1-t) \cdot \psi_{-K_{\mathbb{V}(\mathcal{E}^\vee)}}(\underline{b}') = -t \sum_{k=1}^{n+c} \lambda_k - (1-t) \sum_{k=1}^{n+c} \mu_k \geq -(t+1-t) = -1. \quad (46)$$

As $\underline{x}' \in \text{Supp}(\Sigma')$, there is a cone $\langle \underline{a}_{\tilde{i}_1}, \dots, \underline{a}_{\tilde{i}_n} \rangle \in \Sigma(n)$ such that

$$\underline{x}' = \sum_{k=1}^n \kappa_k \underline{a}'_{\tilde{i}_k} + \sum_{j=1}^c \kappa_{n+j} \underline{e}'_{n+j}$$

for $\kappa_k \in \mathbb{R}_{\geq 0}$. However, we have $\sum_{k=1}^{n+c} \kappa_k > 1$ for otherwise \underline{x}' would lie in $W(\Sigma')$. But $\psi_{-K_{\mathbb{V}(\mathcal{E}^\vee)}}(\underline{x}') = -\sum_{k=1}^{n+c} \kappa_k$, and hence this quantity is strictly smaller than -1 , which contradicts inequality (46), i.e., the convexity of the function $\psi_{-K_{\mathbb{V}(\mathcal{E}^\vee)}}$. \square

Proof of the proposition. We will give a proof of the first statement, the second one is then an easy consequence and will be discussed at the end.

The first part of the argument is parallel to the proof of [RS10, Proposition 2.1 (2)], but using the more precise discussion in the previous two lemmata. From equations (43) and (45) we obtain that

$$C(A'') = \bigcup_{\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle \in \Sigma(n)} C(\underline{a}_0'', \underline{a}_{i_1}'', \dots, \underline{a}_{i_{n+c}}''). \quad (47)$$

This shows the normality of $\mathbb{N}A''$: Given any vector $\underline{x}'' = (x_0, x_1, \dots, x_m, x_{m+1}, \dots, x_{m+c}) \in C(A'') \cap N''$, then there is a maximal cone $\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle$ in $\Sigma(n)$ such that $\underline{x}'' \in C(\underline{a}_0'', \underline{a}_{i_1}'', \dots, \underline{a}_{i_{n+c}}'')$. Hence we have an equation

$$\underline{x}'' = \lambda_0 \underline{a}_0'' + \sum_{k=1}^{n+c} \lambda_k \underline{a}_{i_k}'' \quad (48)$$

with $\lambda_k \in \mathbb{R}_{\geq 0}$ for $k = 0, 1, \dots, n+c$ and $\lambda_k = 0$ for all $k \in \{1, \dots, m\} \setminus \{i_1, \dots, i_n\}$. We know that $(\underline{a}'_{i_1}, \dots, \underline{a}'_{i_{n+c}}) = (\underline{a}'_{i_1}, \dots, \underline{a}'_{i_n}, \underline{e}_{m+1}, \dots, \underline{e}_{m+c})$ is a \mathbb{Z} -basis of N' as $\langle \underline{a}'_{i_1}, \dots, \underline{a}'_{i_{n+c}} \rangle$ is a smooth $n+c$ -dimensional cone in Σ' . It follows that $\underline{a}_0'', \underline{a}_{i_1}'', \dots, \underline{a}_{i_{n+c}}''$ is a \mathbb{Z} -basis of N'' , hence $\lambda_k \in \mathbb{N}$ for $k = 0, 1, \dots, n+c$, and $\underline{x}'' \in \mathbb{N}A''$, which is the defining property of normality of $\mathbb{N}A''$. As usual it follows that $\mathbb{C}[\mathbb{N}A'']$ is Cohen-Macaulay by Hochster's theorem ([Hoc72, Theorem 1]).

It remains to show the second statement concerning the characterization of the interior points of $\mathbb{N}A''$. We will actually show the following

Claim: Let $\underline{x}'' \in \mathbb{N}A''$. Consider the representation (48) of \underline{x}'' as an element of $C(\underline{a}_0'', \underline{a}_{i_1}'', \dots, \underline{a}_{i_{n+c}}'')$, that is, an equation $\underline{x}'' = \sum_{i=0}^{m+c} \lambda_i \underline{a}_i'' \in \mathbb{N}A''$, where $\lambda_k = 0$ if $k \in \{1, \dots, m\} \setminus \{i_1, \dots, i_n\}$. Then \underline{x}'' lies in $(\mathbb{N}A'')^\circ$ iff $\lambda_i > 0$ for $i \in \{0, m+1, \dots, m+c\} = \{0, i_{n+1}, \dots, i_{n+c}\}$.

Notice that a representation as in the claim is unique, if there are two maximal cones of $\Sigma(n)$ such that \underline{x}'' is contained in both of the cones generated by the corresponding column vectors of A'' , then it lies on a common boundary, and the two expressions (48) are equal.

The claim implies that the map Ψ from the proposition is well-defined and surjective, and it is obviously injective. In order to show the claim, notice that

$$(\mathbb{N}A'')^\circ = (C(A'') \setminus \partial C(A'')) \cap N'' = (C(A'') \cap N'') \setminus (\partial C(A'') \cap N'') = \mathbb{N}A'' \setminus (\partial C(A'') \cap N''),$$

so that we have to show that the points in $\partial C(A'') \cap N''$ are precisely those from $\mathbb{N}A''$ where in the above representation (48) there is at least one index $i \in \{0, m+1, \dots, m+c\}$ with $\lambda_i = 0$. From Formula (47) we deduce that

$$\partial C(A'') \subset \bigcup_{\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle \in \Sigma_A(n)} \partial C(\underline{a}_0'', \underline{a}_{i_1}'', \dots, \underline{a}_{i_n}'', \underline{a}_{m+1}'', \dots, \underline{a}_{m+c}'').$$

More precisely, for each $\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle \in \Sigma(n)$ the cone $C(\underline{a}_0'', \underline{a}_{i_1}'', \dots, \underline{a}_{i_n}'', \underline{a}_{m+1}'', \dots, \underline{a}_{m+c}'')$ has two types of facets: those that are facets of $\partial C(\underline{a}_0'', \underline{a}_{i_1}'', \dots, \underline{a}_{m+c}'')$ (call them ‘‘outer boundary’’) and those which are not (‘‘inner boundary’’). The union (over all n -dimensional cones of Σ) of the outer boundaries is the set $\partial C(\underline{a}_0'', \underline{a}_{i_1}'', \dots, \underline{a}_{i_{n+c}}'')$ we are interested in. Moreover, for any subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, m+c\}$ we have that

$$\partial C(\underline{a}_0'', \underline{a}_{i_1}'', \dots, \underline{a}_{i_k}'') = \bigcup_{\substack{\underline{x}' \in \partial \text{Conv}(\underline{0}, \underline{a}'_{i_1}, \dots, \underline{a}'_{i_k}) \\ \lambda \in \mathbb{R}_{\geq 0}}} \lambda \cdot (1, \underline{x}'). \quad (49)$$

In particular,

$$\partial C(A'') = \bigcup_{\substack{\underline{x}' \in \partial \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_m, \underline{e}_{n+1}, \dots, \underline{e}_{n+c}) \\ \lambda \in \mathbb{R}_{\geq 0}}} \lambda \cdot (1, \underline{x}')$$

and we have

$$\partial \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_m, \underline{e}_{n+1}, \dots, \underline{e}_{n+c}) \subset \bigcup_{\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle \in \Sigma(n)} \partial \text{Conv}(\underline{0}, \underline{a}'_{i_1}, \dots, \underline{a}'_{i_n}, \underline{e}_{n+1}, \dots, \underline{e}_{n+c})$$

with a description of $\partial \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_m, \underline{e}_{n+1}, \dots, \underline{e}_{n+c})$ similar to the one above as the union of the “outer boundaries” of all $\partial \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \underline{e}_{n+c})$.

We deduce from the fact that Σ' is smooth (simplicial is actually sufficient) that we have the following decomposition

$$\begin{aligned} \partial \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_{n+c}) &= \partial \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \underline{e}_{n+c}) \\ &\stackrel{!}{=} \text{Conv}(\underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \underline{e}_{n+c}) \cup \bigcup_{k=1}^n \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \widehat{\underline{a}'_k}, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \underline{e}_{n+c}) \\ &\quad \cup \bigcup_{l=1}^c \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{m+1}, \dots, \widehat{\underline{e}_{m+l}}, \dots, \underline{e}_{m+c}). \end{aligned}$$

The facet $\text{Conv}(\underline{0}, \underline{a}'_1, \dots, \widehat{\underline{a}'_k}, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \underline{e}_{n+c})$ is an inner boundary, i.e., it is not contained in $\partial \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{m+1}, \dots, \underline{e}_{m+c})$. This is a consequence of the completeness of Σ , namely, there is some other cone $\langle \underline{a}_{j_1}, \dots, \underline{a}_{j_n} \rangle \in \Sigma$ having $\langle \underline{a}_{i_1}, \dots, \widehat{\underline{a}_{i_k}}, \dots, \underline{a}_{i_n} \rangle$ as a facet, and then similarly the cone $\text{Conv}(\underline{0}, \underline{a}'_1, \dots, \widehat{\underline{a}'_k}, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \underline{e}_{n+c})$ is a facet of both $\text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \underline{e}_{n+c})$ and $\text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \underline{e}_{n+c})$, hence it is not contained in $\partial \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_{m+c})$.

However, both $\text{Conv}(\underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \underline{e}_{n+c})$ and $\text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \widehat{\underline{e}_{n+l}}, \dots, \underline{e}_{n+c})$ (for $l = 1, \dots, c$) are facets of $\text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_{m+c})$, i.e., they are outer boundaries. We conclude that

$$\begin{aligned} \partial \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_{m+c}) &= \bigcup_{\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle \in \Sigma(n)} \left[\text{Conv}(\underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \underline{e}_{n+c}) \cup \right. \\ &\quad \left. \bigcup_{l=1}^c \text{Conv}(\underline{0}, \underline{a}'_1, \dots, \underline{a}'_n, \underline{e}_{n+1}, \dots, \widehat{\underline{e}_{n+l}}, \dots, \underline{e}_{n+c}) \right]. \end{aligned}$$

Using Equation (49), this gives

$$\begin{aligned} \partial C(A'') &= \partial C(\underline{a}''_0, \underline{a}''_1, \dots, \underline{a}''_{m+c}) \stackrel{!}{=} \bigcup_{\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle \in \Sigma(n)} \left[C(\underline{a}''_1, \dots, \underline{a}''_n, \underline{a}''_{m+1}, \dots, \underline{a}''_{m+c}) \cup \right. \\ &\quad \left. \bigcup_{l=1}^c C(\underline{a}''_0, \underline{a}''_1, \dots, \underline{a}''_n, \underline{a}''_{m+1}, \dots, \widehat{\underline{a}''_{m+l}}, \dots, \underline{a}''_{m+c}) \right]. \end{aligned}$$

Now we see that for any point $\partial C(A'') \cap N$, either the coefficient λ_0 or the coefficient λ_{m+l} for some $l \in \{1, \dots, c\}$ in the representation (48) is necessarily zero. This shows the claim, and proves that the map Ψ is an isomorphism. Finally, it follows from standard arguments about semigroup rings (see, e.g. [BH93, corollary 6.3.8]) that $\mathbb{C}[\mathbb{N}A'']$ is Gorenstein.

We now show the second statement. First we deduce from the above arguments that we also have $C(A') \cap N' = \mathbb{N}A'$, that is, that $\mathbb{N}A'$ is normal. Namely, consider the projection $p : N''_{\mathbb{R}} \rightarrow N'_{\mathbb{R}}$ forgetting the first component. Applying p to both sides of equation (47) it follows that

$$C(A') = \bigcup_{\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle \in \Sigma(n)} C(\underline{a}'_1, \dots, \underline{a}'_{n+c}).$$

On the other hand, the smoothness of the fan Σ' gives that $C(\underline{a}'_1, \dots, \underline{a}'_{n+c}) \cap N' = \sum_{j=1}^{n+c} \mathbb{N}a'_{i_j}$ for any $\underline{a}_{i_1}, \dots, \underline{a}_{i_{n+c}}$ such that $\langle \underline{a}_{i_1}, \dots, \underline{a}_{i_n} \rangle \in \Sigma(n)$. In conclusion, we obtain that $C(A') \cap N' = \mathbb{N}A'$, that is, $\mathbb{N}A'$ is normal. In particular, we have $(\mathbb{N}A')^\circ = (C(A') \setminus \partial C(A')) \cap N'$. From the definition of the matrix A'' we see that $p(C(A'')) = C(A')$. We claim that

$$p(C(A'') \setminus \partial C(A'')) = C(A') \setminus \partial C(A'). \quad (50)$$

This can be seen as follows: For any point $\underline{x}'' \in C(A'') \setminus \partial C(A'')$, there exists a small ε such that

$$B_\varepsilon^{|\cdot|}(\underline{x}'') \subset C(A''), \quad (51)$$

where $|\cdot|^{\max}$ is the maximum norm on $N_{\mathbb{R}}'' = \mathbb{R}^{m+s+1}$ and $B_{\varepsilon}^{|\cdot|^{\max}}(\underline{x}'')$ is the open $n+c+1$ -dimensional hypercube with ‘‘radius’’ ε . Put $\underline{x}' := p(\underline{x}'')$, then we have $p(B_{\varepsilon}^{|\cdot|^{\max}}(\underline{x}'')) = B_{\varepsilon}^{|\cdot|^{\max}}(\underline{x}')$ and equation (51) shows that this $n+c$ -dimensional hypercube is contained in $C(A')$. Hence $\underline{x}' \in C(A') \setminus \partial C(A')$ so that $p(C(A'') \setminus \partial C(A'')) \subset C(A') \setminus \partial C(A')$. For the inverse inclusion, take $\underline{x}' \in C(A') \setminus \partial C(A')$, then again there is an $n+c$ -dimensional hypercube $B_{\varepsilon}^{|\cdot|^{\max}}(\underline{x}')$ contained in $C(A')$. By construction, the cone

$$C\left(\{(1, \underline{x}') \in \{1\} \times N_{\mathbb{R}}' \mid \underline{x}' \in B_{\varepsilon}^{|\cdot|^{\max}}(\underline{x}')\}\right)$$

over this hypercube is contained in $C(A'')$. Then there must be some $\varepsilon' \leq \varepsilon$ such that

$$B_{\varepsilon'}^{|\cdot|^{\max}}(\underline{x}'') \subset C\left(\{(1, \underline{x}') \in \{1\} \times N_{\mathbb{R}}' \mid \underline{x}' \in B_{\varepsilon}^{|\cdot|^{\max}}(\underline{x}')\}\right) \subset C(A''),$$

where $\underline{x}'' := (1, \underline{x}')$, so that finally $\underline{x}'' \in C(A'') \setminus \partial C(A'')$ and the equality (50) holds.

On the other hand, the definition of the matrix A'' shows that

$$p(C(A'') \setminus \partial C(A'')) = p(\underline{a}_0'' + \underline{a}_{m+1}'' + \dots + \underline{a}_{m+c}'' + C(A'')) = \underline{a}_{m+1}' + \dots + \underline{a}_{m+c}' + C(A')$$

so that we obtain

$$(\mathbb{N}A')^{\circ} = \sum_{j=1}^c \underline{a}_j' + \mathbb{N}A'$$

as required. □

As a consequence, we obtain the following duality result for those GKZ systems that we will be interested in the sequel.

Theorem 5.4. *Let A'' as above, that is, suppose that its columns $(\underline{a}_0'', \underline{a}_1'', \dots, \underline{a}_{m+c}'')$ are of the form $\underline{a}_i'' = (1, \underline{a}_i')$ where $\underline{a}_0'' = (1, \underline{0})$ and where \underline{a}_i' ($i = 1, \dots, m+c$) are the integral primitive generator of the fan of $\mathbb{V}(\mathcal{E}^{\vee})$. For $\beta \in \mathbb{Z}^{1+m+c}$, consider the GKZ-system $\mathcal{M}_{A''}^{\beta}$ as in Definition 2.7.*

1. *There is an isomorphism*

$$\mathbb{D}(\mathcal{M}_{A''}^{(0,0,0)}) \cong \mathcal{M}_{A''}^{-(c+1,0,1)} = \mathcal{M}_{A''}^{-\underline{a}_0'' - \sum_{i=1}^c \underline{a}_{m+i}'},$$

and we call the map

$$\phi : \mathcal{M}_{A''}^{-(c+1,0,1)} \longrightarrow \mathcal{M}_{A''}^{(0,0,0)}$$

induced by right multiplication by $\partial_0 \cdot \partial_{m+1} \cdot \dots \cdot \partial_{m+c}$ the duality morphism. For any $\beta_0 \in \mathbb{Z}$, we obtain an induced morphism

$$\widehat{\phi} : \widehat{\mathcal{M}}_{A'}^{(\beta_0, 0, -1)} \longrightarrow \widehat{\mathcal{M}}_{A'}^{(\beta_0 + c, 0, 0)}$$

given by right multiplication with $\partial_{m+1} \cdot \dots \cdot \partial_{m+c}$ (see 3.1 for the definition of the modules $\widehat{\mathcal{M}}^{\beta}$). The case $\beta_0 = -2c$ will be particularly important, and we will also call the map

$$\widehat{\phi} : \widehat{\mathcal{M}}_{A'}^{-(2c, 0, 1)} \longrightarrow \widehat{\mathcal{M}}_{A'}^{(-c, 0, 0)}$$

the duality morphism.

2. *Consider the natural good filtration $F_{\bullet} \mathcal{M}_{A''}^{\beta}$ induced by the order filtration on \mathcal{D} . Let $\mathbb{D}(\mathcal{M}_{A''}^{\beta}, F_{\bullet})$ be the dual filtered module in the sense of [Sai88, section 2.4], i.e., $\mathbb{D}(\mathcal{M}_{A''}^{\beta}, F_{\bullet}) = (\mathbb{D}\mathcal{M}_{A''}^{\beta}, F_{\bullet}^{\mathbb{D}})$ where $F_{\bullet}^{\mathbb{D}}(\mathbb{D}\mathcal{M}_{A''}^{\beta})$ is the filtration dual to $F_{\bullet} \mathcal{M}_{A''}^{\beta}$. Then we have*

$$\mathbb{D}\left(\mathcal{M}_{A''}^{-\underline{a}_0'' - \sum_{i=1}^c \underline{a}_{m+i}'}, F_{\bullet}\right) \cong (\mathcal{M}_{A''}^{(0,0,0)}, F_{\bullet+n-(m+c+1)}).$$

Proof. 1. The proof is parallel to [Wal07, Proposition 4.1] or [RS10, Theorem 2.15 and Proposition 2.18], so that we only sketch it here, referring to loc.cit. for details. First one has to define the so-called Euler-Koszul complex resp. co-complex (see [MMW05]). Its global sections complex $K_{\bullet}(T, E - \beta)$ is a complex of free $D_V \otimes_R T$ -modules where $R = \mathbb{C}[\partial_0, \partial_1, \dots, \partial_{m+c}]$ and where T is a so-called *toric* R -module. A particular case is $T = \mathbb{C}[\mathbb{N}A'']$. Notice that the terms of $K_{\bullet}(T, E - \beta)$ are not free over D_V . However, for $T = \mathbb{C}[\mathbb{N}A'']$, this complex is a resolution by left D_V -modules of the modules $M_{A''}^{\beta}$. The differentials of $K_{\bullet}(T, E - \beta)$ are defined by the operators E and Z_k entering in the definition of $M_{A''}^{\beta}$. From a resolution of the toric ring $\mathbb{C}[\mathbb{N}A'']$ by free $\mathbb{C}[\partial_0, \partial_1, \dots, \partial_{m+c}]$ -modules one can also construct a resolution of $M_{A''}^{(0, \underline{0}, \underline{0})}$ by free D_V -modules. Applying $\text{Hom}_{D_V}(-, D_V)$ yields basically the same complex, but where the parameters in the differentials are changed, and where the toric module is now the canonical module of the ring $\mathbb{C}[\mathbb{N}A'']$. Now from the Gorenstein property of $\mathbb{C}[\mathbb{N}A'']$ with the precise description of the interior ideal from Proposition 5.1 we obtain the desired result by taking the cohomology of the two complexes, that is, we can show the identification of the holonomic dual of $\mathcal{M}_{A''}^{(0, \underline{0}, \underline{0})}$ with $\mathcal{M}_{A''}^{-(c+1, \underline{0}, \underline{1})}$.

2. The proof is literally the same as in [RS10, Proposition 2.19, 2.] with the indices shifted appropriately. □

As a consequence, we can make more specific statements on the parameter vectors of the various GKZ-systems occurring in the results of the previous sections.

Corollary 5.5. *Consider the situation in section 2 where the matrix B is A' , i.e., given by the primitive integral generators of the fan of $\mathbb{V}(\mathcal{E}^{\vee})$, in particular, both $\mathbb{N}B = \mathbb{N}A'$ and $\mathbb{N}B = \mathbb{N}A''$ are normal semigroups. Then*

1. *The statements of Theorem 2.11, Theorem 2.16 and of Proposition 2.17 hold true for the parameter values $\beta = (0, \underline{0}, \underline{0})$, $\beta' = (c+1, \underline{0}, \underline{1}) \in \mathbb{Z}^{1+n+c}$.*
2. *The statements of Proposition 3.3 and of Theorem 3.6 hold true for the parameter values $\beta = (0, \underline{0})$, $\beta' = (\underline{0}, \underline{1}) \in \mathbb{Z}^{n+c}$ and for any $\beta_0, \beta'_0 \in \mathbb{Z}$.*

Remark: In our previous paper [RS10], we obtained from a similar construction a non-degenerate pairing on the Fourier-Laplace transformed GKZ-system (see [RS10, corollary 2.20], where this system was called $\widehat{\mathcal{M}}_{\tilde{A}}$). It was given by an isomorphism of $\widehat{\mathcal{M}}_{\tilde{A}}$ to its holonomic dual (which is isomorphic to its meromorphic dual, see also the proof of Lemma 6.4, 3. below). The fact that in the current situation, we only have a morphism $\widehat{\phi} : \widehat{\mathcal{M}}_{A'}^{-(2c, \underline{0}, \underline{1})} \rightarrow \widehat{\mathcal{M}}_{A'}^{(-c, \underline{0}, \underline{0})}$ which is not an isomorphism unless $c = 0$ (in which case we are exactly in the situation of [RS10], see the remark at the end of section 6 of this paper) corresponds to the fact that the pairing S on the twisted quantum \mathcal{D} -module as introduced in Definition 4.3 is degenerate. As we have seen in the definition of the reduced quantum- \mathcal{D} -module, it becomes non-degenerate when we divide out the kernel of the cup product with the first Chern classes of the line bundles \mathcal{L}_j . We will show below in corollary 6.12 that the reduced quantum \mathcal{D} -module is part of a *non-commutative* Hodge structure, which implies in particular that it carries a non-degenerate pairing like the one from [RS10].

6 Mirror correspondences

In this section we combine the results obtained so far with the GKZ-type description of the ambient resp. reduced quantum \mathcal{D} -modules from [MM11] for the toric case. We obtain a mirror statement which identifies them with \mathcal{D} -modules constructed from our Landau-Ginzburg models. The results from section 2 will be applied for the case where the matrix B (used for the construction of GKZ-systems and of families of Laurent polynomials) is given by A' (see Definition 4.6) the columns of which are the primitive integral generators of the fan of the total bundle $\mathbb{V}(\mathcal{E}^{\vee})$. Recall also (remark at the end of section 4) that we denote by A'' the matrix constructed from A' by adding 1 as an extra component to all columns and by adding $(1, \underline{0})$ as extra column. Hence, if B is equal to A' , then the matrix \widetilde{B} used in

section 2 is exactly the matrix A'' . Recall also that the parameter of the GKZ-systems of the matrix A'' is written as $(\alpha, \underline{\gamma}, \underline{\delta}) \in \mathbb{C}^{1+m+c}$ with $\alpha \in \mathbb{C}$, $\underline{\gamma} \in \mathbb{C}^m$ and $\underline{\delta} \in \mathbb{C}^c$.

The starting point for our discussion here is the duality morphism from the last section. We need to consider a slight variation of it, which is defined only outside the boundary $\lambda_i = 0$ and only outside the bad parameter locus as defined in subsection 3.2. Recall that $V = \mathbb{C}_{\lambda_0} \times W$, and that this bad parameter locus of the family $\varphi_{A'}$ was called $W^{bad} \subset W$. The complement of this locus outside the boundary $\lambda_i = 0$ was called W° , that is, $W^\circ := W^* \setminus W^{bad}$.

Definition-Lemma 6.1. *For any $\beta = (\beta_0, \beta_1, \dots, \beta_m, \beta_{m+1}, \dots, \beta_{n+c}) \in \mathbb{Z}^{1+n+c}$, consider the GKZ-system $\mathcal{M}_{A'}^\beta$, as well as the Fourier-Laplace transformed version $\widehat{\mathcal{M}}_{A'}^\beta$ as introduced in Definition 3.1. Let j denote the inclusion $W^* \hookrightarrow W$. Then we put ${}^*\widehat{\mathcal{M}}_{A'}^\beta := (\text{id}_{\mathbb{C}_z} \times j)^+ \widehat{\mathcal{M}}_{A'}^\beta$, resp. ${}^0\widehat{\mathcal{M}}_{A'}^\beta := (\text{id}_{\mathbb{C}_z} \times j)^* ({}^0\widehat{\mathcal{M}}_{A'}^\beta)$, that is, we have*

$${}^*\widehat{\mathcal{M}}_{A'}^\beta = \frac{\mathcal{D}_{\mathbb{C}_z \times W^*}[z^{-1}]}{\mathcal{D}_{\mathbb{C}_z \times W^*}[z^{-1}](\widehat{\square}_l^*)_{l \in \mathbb{L}_{A'}} + \mathcal{D}_{\mathbb{C}_z \times W^*}[z^{-1}](\widehat{E}_k - z\beta_k)_{k=0, \dots, n+c}},$$

where

$$\widehat{\square}_l^* := \prod_{i \in \{1, \dots, m+c\}: l_i > 0} \lambda_i^{l_i} (z \cdot \partial_i)^{l_i} - \prod_{i=1}^{m+c} \lambda_i^{l_i} \cdot \prod_{i \in \{1, \dots, m+c\}: l_i < 0} \lambda_i^{-l_i} (z \cdot \partial_i)^{-l_i},$$

$$\widehat{E}_0 := z^2 \partial_z + \sum_{i=1}^{m+c} \lambda_i \cdot z \partial_i,$$

$$\widehat{E}_k := \sum_{i=1}^{m+c} a'_{ki} \lambda_i \cdot z \partial_i \quad \forall k = 1, \dots, n+c$$

and moreover, ${}^0\widehat{\mathcal{M}}_{A'}^\beta$ is the $\mathcal{R}_{\mathbb{C}_z \times W^*}$ -subalgebra generated by [1], and we have

$${}^0\widehat{\mathcal{M}}_{A'}^\beta = \frac{\mathcal{R}_{\mathbb{C}_z \times W^*}}{\mathcal{R}_{\mathbb{C}_z \times W^*}(\widehat{\square}_l^*) + \mathcal{R}_{\mathbb{C}_z \times W^*}(\widehat{E}_k - z\beta_k)_{k=0, \dots, n+c}}.$$

Moreover, we define the modules ${}^*\widehat{\mathcal{N}}_{A'}^\beta$ as the cyclic quotients of $\mathcal{D}_{\mathbb{C}_z \times W^*}[z^{-1}]$ by the left ideal generated by $\widehat{\square}_l$ for $l \in \mathbb{L}_{A'}$ and $\widehat{E}_k - z\beta_k$ for $k = 0, \dots, n+c$, where

$$\begin{aligned} \widehat{\square}_l := & \prod_{i \in \{1, \dots, m\}: l_i > 0} \lambda_i^{l_i} (z \cdot \partial_i)^{l_i} \prod_{i \in \{m+1, \dots, m+c\}: l_i > 0} \prod_{\nu=1}^{l_i} (\lambda_i (z \cdot \partial_i) - z \cdot \nu) \\ & - \prod_{i=1}^{m+c} \lambda_i^{l_i} \cdot \prod_{i \in \{1, \dots, m\}: l_i < 0} \lambda_i^{-l_i} (z \cdot \partial_i)^{-l_i} \prod_{i \in \{m+1, \dots, m+c\}: l_i < 0} \prod_{\nu=1}^{-l_i} (\lambda_i (z \cdot \partial_i) - z \cdot \nu). \end{aligned}$$

Consider the morphism

$$\Psi : {}^*\widehat{\mathcal{N}}_{A'}^{(0,0,0)} \longrightarrow {}^*\widehat{\mathcal{M}}_{A'}^{-(2c,0,1)} \quad (52)$$

given by right multiplication with $z^c \cdot \prod_{i=m+1}^{m+c} \lambda_i$. As it is obviously invertible, the two modules ${}^*\widehat{\mathcal{N}}_{A'}^{(0,0,0)}$ and ${}^*\widehat{\mathcal{M}}_{A'}^{-(2c,0,1)}$ are isomorphic. We define $\widetilde{\phi}$ to be the composition $\widetilde{\phi} := \widehat{\phi} \circ \Psi$, where $\widehat{\phi}$ is the duality morphism from Theorem 5.4. In concrete terms, we have:

$$\widetilde{\phi} : {}^*\widehat{\mathcal{N}}_{A'}^{(0,0,0)} \longrightarrow {}^*\widehat{\mathcal{M}}_{A'}^{(-c,0,0)},$$

$$m \longmapsto \widehat{\phi}(m \cdot z^c \cdot \lambda_{m+1} \cdot \dots \cdot \lambda_{m+c}) = m \cdot (z\lambda_{m+1}\partial_{m+1}) \cdot \dots \cdot (z\lambda_{m+c}\partial_{m+c}).$$

In view of corollary 5.5, 2., we obtain

$$\text{im}(\widetilde{\phi}) \cong \text{im}(\widehat{\phi}) \cong (\text{id}_{\mathbb{C}_z} \times j)^+ \widehat{\mathcal{M}}^{IC}(X^\circ, \mathcal{L}). \quad (53)$$

For any $\beta \in \mathbb{Z}^{1+n+c}$, consider the $\mathcal{R}_{\mathbb{C}_z \times W^*}$ -subalgebra of

$$\mathcal{D}_{\mathbb{C}_z \times W^*}[z^{-1}] / \mathcal{D}_{\mathbb{C}_z \times W^*}[z^{-1}] \left((\widehat{\square}_l)_{l \in \mathbb{L}_{A'}} \right) + \mathcal{D}_{\mathbb{C}_z \times W^*}[z^{-1}] \left(\widehat{E}_k - z\beta_k \right)_{k=0, \dots, n+c}$$

generated by the element [1] and denote its restriction to $\mathbb{C}_z \times W^\circ$ by ${}^0\widehat{\mathcal{N}}_{A'}^\beta$. Similarly to corollary 3.19, we have

$${}^0\widehat{\mathcal{N}}_{A'}^\beta = \left[\frac{\mathbb{C}[z, \lambda_1^\pm, \dots, \lambda_{m+c}^\pm] \langle z^2 \partial_z, z \partial_{\lambda_1}, \dots, z \partial_{\lambda_{m+c}} \rangle}{\left((\widehat{\square}_l \right)_{l \in \mathbb{L}_{A'}}, (\widehat{E}_k - z \cdot \beta_k)_{k=0, \dots, n+c} \right)} \right]_{|\mathbb{C}_z \times W^\circ}.$$

In the next lemma we want to describe the restriction of the \mathcal{D} -module $\widehat{\mathcal{M}}^{IC}(X^\circ, \mathcal{L})$ to $\mathbb{C}_z \times W^*$.

Lemma 6.2. *Consider the morphism $\widehat{\phi} : \widehat{\mathcal{M}}_{A'}^{-(2c, 0, \underline{1})} \rightarrow \widehat{\mathcal{M}}_{A'}^{(-c, 0, \underline{0})}$ from Theorem 5.4 and the isomorphisms $\widehat{\mathcal{M}}^{IC}(X^\circ, \mathcal{L}) \simeq \text{im}(\widehat{\phi}) \simeq \widehat{\mathcal{M}}_{A'}^{-(2c, 0, \underline{1})} / \ker(\widehat{\phi})$ from Corollary 5.5. We have the following isomorphism*

$$(id_{\mathbb{C}_z} \times j)^+ \widehat{\mathcal{M}}^{IC}(X^\circ, \mathcal{L}) \simeq {}^* \widehat{\mathcal{M}}_{A'}^{-(2c, 0, \underline{1})} / \widehat{\mathcal{K}}_{\mathcal{M}} \simeq {}^* \widehat{\mathcal{N}}_{A'}^{(0, 0, \underline{0})} / \widehat{\mathcal{K}}_{\mathcal{N}},$$

where $\widehat{\mathcal{K}}_{\mathcal{M}}$ resp. $\widehat{\mathcal{K}}_{\mathcal{N}}$ are the sub- \mathcal{D} -modules associated to the sub- \mathcal{D} -modules

$$\{m \in {}^* \widehat{\mathcal{M}}^{-(2c, 0, \underline{1})} \mid \exists p \in \mathbb{Z}, k \in \mathbb{N} \text{ such that } (\lambda \partial + p) \dots (\lambda \partial + p + k)m = 0\}$$

resp.

$$\{n \in {}^* \widehat{\mathcal{N}}^{(0, 0, \underline{0})} \mid \exists p \in \mathbb{Z}, k \in \mathbb{N} \text{ such that } (\lambda \partial + p) \dots (\lambda \partial + p + k)n = 0\}$$

with $(\lambda \partial + i) := \prod_{j=m+1}^{m+c} (\lambda_j \partial_j + i)$ for $i \in \mathbb{Z}$.

Proof. We will first compute the restriction of $\mathcal{M}^{IC}(X^\circ, \mathcal{L})$ to $V^* = \mathbb{C}_{\lambda_0} \times W^*$. Recall the morphism $\phi : M_{A''}^{-(c+1, 0, \underline{1})} \rightarrow M_{A''}^{(0, 0, \underline{0})}$ from Theorem 5.4. We have $M^{IC}(X^\circ, \mathcal{L}) \simeq M_{A''}^{-(c+1, 0, \underline{1})} / \ker(\phi)$ where $\ker(\phi)$ is given by

$$\{m \in M_{A''}^{-(c+1, 0, \underline{1})} \mid \exists n \in \mathbb{N} \text{ such that } (\partial_0 \cdot \partial_{m+1} \dots \partial_{m+c})^n m = 0\}.$$

Notice that $\mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} M^{IC}(X^\circ, \mathcal{L}) \simeq {}^* M_{A''}^{-(c+1, 0, \underline{1})} / (\mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} \ker(\phi))$, where ${}^* M_{A''}^{-(c+1, 0, \underline{1})}$ is the module of global sections of ${}^* \mathcal{M}_{A''}^{-(c+1, 0, \underline{1})}$ and $\mathbb{C}[\lambda^\pm]$ resp. $\mathbb{C}[\lambda]$ is shorthand for $\mathbb{C}[\lambda_0, \dots, \lambda_m, \lambda_{m+1}^\pm, \dots, \lambda_{m+c}^\pm]$ resp. $\mathbb{C}[\lambda_0, \dots, \lambda_m, \lambda_{m+1}, \dots, \lambda_{m+c}]$.

We want to characterize $\mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} \ker(\phi)$ inside ${}^* M_{A''}^{-(c+1, 0, \underline{1})} = \mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} M_{A''}^{-(c+1, 0, \underline{1})}$. For this we define the following submodule in ${}^* M_{A''}^{-(c+1, 0, \underline{1})}$:

$$K := \{m \in {}^* M_{A''}^{-(c+1, 0, \underline{1})} \mid \exists p \in \mathbb{Z}, k \in \mathbb{N} \text{ such that } \partial_0^{k+1} (\lambda \partial + p) \dots (\lambda \partial + p + k)m = 0\}.$$

Consider the following element of $\mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} \ker(\phi)$:

$$\lambda_{m+1}^{-p_1} \dots \lambda_{m+c}^{-p_c} \otimes m \quad \text{with } p_1, \dots, p_c \in \mathbb{N}, \quad (54)$$

i.e. there exists an $n \in \mathbb{N}$ such that $(\partial_0 \cdot \partial_{m+1} \dots \partial_{m+c})^{n+1} m = 0$. Therefore we have

$$\begin{aligned} 0 &= \lambda_{m+1}^{-p_1} \dots \lambda_{m+c}^{-p_c} \otimes (\partial_0 \cdot \partial_{m+1} \dots \partial_{m+c})^{n+1} m \\ &= \lambda_{m+1}^{-p_1} \dots \lambda_{m+c}^{-p_c} \otimes (\lambda_{m+1} \dots \lambda_{m+c})^{n+1} (\partial_0 \cdot \partial_{m+1} \dots \partial_{m+c})^{n+1} m \\ &= \partial_0^{n+1} \cdot (\lambda_{m+1}^{-p_1} \dots \lambda_{m+c}^{-p_c} \otimes (\lambda \partial) \dots (\lambda \partial - n)m) \\ &= \partial_0^{n+1} (\lambda \partial + p_{max}) \dots (\lambda \partial + p_{min} - n) \cdot (\lambda_{m+1}^{-p_1} \dots \lambda_{m+c}^{-p_c} \otimes m) \\ &= \partial_0^{k+1} (\lambda \partial + p) \dots (\lambda \partial + p + k) \cdot (\lambda_{m+1}^{-p_1} \dots \lambda_{m+c}^{-p_c} \otimes m), \end{aligned}$$

where $p_{max} := \max\{p_i\}$, $p_{min} := \min\{p_i\}$, $p := p_{min} - n$ and $k := p_{max} - p_{min} + n$. Because $\mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} \ker(\phi)$ is generated by elements of the form (54), we see that $\mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} \ker(\phi) \subset K$. Therefore we have a surjective morphism

$$\mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} M^{IC}(X^\circ, \mathcal{L}) \simeq {}^* M_{A''}^{-(c+1, 0, \underline{1})} / (\mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} \ker(\phi)) \twoheadrightarrow {}^* M_{A''}^{-(c+1, 0, \underline{1})} / K.$$

Because $\mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} M^{IC}(X^\circ, \mathcal{L})$ corresponds to the restriction of the simple \mathcal{D} -module $\mathcal{M}^{IC}(X^\circ, \mathcal{L})$ to the open subset V^* , it is itself simple. Thus, $*M_{A''}^{-(c+1,0,1)}/K$ is either equal to 0 or is isomorphic to $\mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} M^{IC}(X^\circ, \mathcal{L})$.

In order to prove the lemma, we are going to show that $K \subsetneq *M_{A''}^{-(c+1,0,1)}$. Denote by $F_\bullet *M_{A''}^{-(c+1,0,1)}$ the good filtration on $*M_{A''}^{-(c+1,0,1)}$ which is induced by the order filtration on D_{V^*} . Notice that we have

$$K \subsetneq *M_{A''}^{-(c+1,0,1)} \iff gr^F K \subsetneq gr^F *M_{A''}^{-(c+1,0,1)} \quad (55)$$

In order to show that $gr^F K \subsetneq gr^F *M_{A''}^{-(c+1,0,1)}$, we first remark that

$$gr^F K \subset \{ \bar{m} \in gr^F *M_{A''}^{-(c+1,0,1)} \mid \exists k \in \mathbb{N} \text{ such that } \mu_0^{k+1} \lambda^{k+1} \mu^{k+1} \bar{m} = 0 \},$$

where $\lambda = (\lambda_{m+1} \cdots \lambda_{m+c})$, $\mu = (\mu_{m+1} \cdots \mu_{m+c})$ and μ_i is the symbol $\sigma(\partial_{\lambda_i})$.

Thus, in order to show the right hand side of (55), it is enough to show that $\text{char}(*M_{A''}^{-(c+1,0,1)}) = \text{supp}(gr^F *M_{A''}^{-(c+1,0,1)}) \subset T^*(V^*)$ is not contained in $\{\mu_0 \cdot \underline{\mu} \cdot \underline{\lambda} = 0\}$.

Therefore it is enough to find a vector $(\underline{\mu}', \underline{\lambda}') \in \text{char}(*M_{A''}^{-(c+1,0,1)}) \subset T^*(V^*)$ with $\mu'_0 \cdot \mu' \cdot \lambda' \neq 0$, resp. a vector $(\underline{\mu}', \underline{\lambda}') \in \text{char}(M_{A''}^{-(c+1,0,1)}) \subset T^*(V)$ with $\mu'_0 \cdot \mu' \neq 0$ and $\lambda'_i \neq 0$ for $i = 1, \dots, m+c$.

Notice that we have

$$\text{char}(M_{A''}^{-(c+1,0,1)}) = \text{char}(M_{A''}^{(0,0,0)}) = \text{char}(\text{FL}(M_{A''}^{(0,0,0)})) = \text{char}(h_+ \mathcal{O}_T),$$

where the first equality follows from [GKZ89, Theorem 4], the second equality follows e.g. from [Bry86, Corollaire 7.25] and the third equality follows from [SW09, Corollary 3.7]. Recall that the coordinates on V' are denoted by μ_i for $i = 0, \dots, m+c$ and the symbols of ∂_{μ_i} are denoted by λ_i . We now compute the fiber of $\text{char}(h_+ \mathcal{O}_T) \rightarrow V'$ over the point $\underline{\mu} = (1, \dots, 1)$. Recall that the map

$$\begin{aligned} h : T &\longrightarrow V', \\ (y_0, \dots, y_{m+c}) &\mapsto (\underline{y}^{\alpha''_0}, \dots, \underline{y}^{\alpha''_{m+c}}) \end{aligned}$$

can be factored into a closed embedding $h' : T \rightarrow (\mathbb{C}^*)^{m+c+1}$ and an open embedding $(\mathbb{C}^*)^{m+c+1} \rightarrow V'$. Therefore the fiber of the characteristic variety over $(1, \dots, 1)$ is just the fiber of the conormal bundle of $h'(T)$ in $(\mathbb{C}^*)^{m+c+1}$. The tangent space of $h'(T)$ at $(1, \dots, 1)$ is generated by

$$\sum_{i=0}^{m+c} a''_{ki} \partial_{\mu_i} \quad \text{for } k = 0, \dots, m+c.$$

Therefore $(\underline{1}, \underline{\lambda}')$ lies in $\text{char}(h_+ \mathcal{O}_T)$ if and only if $\sum_{i=0}^{m+c} a''_{ki} \lambda'_i = 0$ for all $k = 0, \dots, m+c$. So it remains to prove that there exists such a $\underline{\lambda}'$ with $\lambda'_i \neq 0$ for $i = 1, \dots, m+c$. First notice that it is enough to construct a $(\lambda_1^\circ, \dots, \lambda_{m+c}^\circ)$ with

$$\sum_{i=1}^{m+c} a'_{ki} \lambda_i^\circ = 0 \quad (56)$$

for all $k = 1, \dots, m+c$ and $\lambda_i^\circ \neq 0$ for all $i = 1, \dots, m+c$. Recall the structure of the matrix A' :

$$A' = \left(\begin{array}{c|c} A & 0_{n,c} \\ \hline (d_{ji}) & \text{Id}_c \end{array} \right), \quad (57)$$

where $d_{ji} \geq 0$ and the columns \underline{a}_i of the matrix A are the primitive integral generators of the rays of the fan Σ corresponding to a complete, smooth toric variety X_Σ . This ensures the existence of $(\lambda_1^\circ, \dots, \lambda_m^\circ) \in \mathbb{Z}_{>0}^m$ with $\sum_{i=1}^m \lambda_i^\circ \underline{a}_i = 0$. Setting $\lambda_{m+j}^\circ := -\sum_{i=1}^m d_{ji} \lambda_i^\circ$, we have constructed an element $(\lambda_1^\circ, \dots, \lambda_{m+c}^\circ)$ with $\lambda_j^\circ \neq 0$ and satisfying $\sum_{i=1}^{m+c} a'_{ki} \lambda_i^\circ = 0$. Summarizing, this shows that $K \subsetneq *M_{A''}^{-(c+1,0,1)}$, i.e.

$$\mathbb{C}[\lambda^\pm] \otimes_{\mathbb{C}[\lambda]} M^{IC}(X^\circ, \mathcal{L}) \simeq *M_{A''}^{-(c+1,0,1)}/K.$$

Applying the localized Fourier-Laplace transformation to this isomorphism, we obtain the first isomorphism in the statement of the lemma. The second isomorphism follows from the \mathcal{D} -linearity of the isomorphism $*\widehat{\mathcal{M}}_{A'}^{-(2c,0,1)} \simeq *\widehat{\mathcal{N}}_{A'}^{(0,0,0)}$. \square

As in [RS10, section 3], we proceed by studying the restriction of the modules ${}^*\mathcal{M}_{A''}^\beta$, ${}^*\widehat{\mathcal{M}}_{A''}^\beta$, and ${}^*\widehat{\mathcal{N}}_{A''}^\beta$ to the Kähler moduli space of $\mathbb{V}(\mathcal{E}^\vee)$ as described in the second part of section 4 (see Lemma 4.4 and Proposition 4.7). The following construction has some overlap with the considerations in subsection 2.4 on which we comment later.

We apply $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the exact sequence (39) to obtain the following exact sequence

$$1 \longrightarrow (\mathbb{C}^*)^{n+c} \longrightarrow (\mathbb{C}^*)^{m+c} \longrightarrow \mathbb{L}_{A'}^\vee \otimes \mathbb{C}^* \longrightarrow 1. \quad (58)$$

We will identify the middle torus with $\text{Spec } \mathbb{C}[\lambda_1^\pm, \dots, \lambda_{m+c}^\pm]$, this space was called W^* in section 2. Choose a basis (p_1, \dots, p_r) of $\mathbb{L}_{A'}^\vee$ with the following properties

1. $p_a \in \mathcal{K}_{\mathbb{V}(\mathcal{E}^\vee)} = \mathcal{K}_{X_{\mathbb{E}}}$ for all $a = 1, \dots, r$,
2. $\sum_{i=1}^{m+c} \overline{D}_i \in \sum_{a=1}^r \mathbb{R}_{\geq 0} p_a$.

Using the basis $(p_a)_{a=1, \dots, r}$, we identify $\mathbb{L}_{A'}^\vee \otimes \mathbb{C}^*$ with $(\mathbb{C}^*)^r$ and obtain coordinates q_1, \dots, q_r on this space. We will write \mathcal{KM} for this space and call it complexified Kähler moduli space. Notice that the choice of coordinates is considered as part of the data of \mathcal{KM} , that is, we really have $\mathcal{KM} = (\mathbb{C}^*)^r$ and not only $\mathcal{KM} = \mathbb{L}_{A'}^\vee \otimes \mathbb{C}^*$.

Consider the embedding $\mathbb{L}_{A'} \hookrightarrow \mathbb{Z}^{m+c}$, which is given by a matrix $L \in \text{Mat}((m+c) \times r, \mathbb{Z})$ with respect to the basis p_a^\vee of $\mathbb{L}_{A'}$ and the natural basis of \mathbb{Z}^{m+c} . Choose a section $\mathbb{Z}^{m+c} \rightarrow \mathbb{L}_{A'}$ of this inclusion, which is given by a matrix $M \in \text{Mat}(r \times (m+c), \mathbb{Z})$. This defines a section on the dual lattices, i.e. a section $\mathbb{L}_{A'}^\vee \rightarrow \mathbb{Z}^{m+c}$ of the projection $\mathbb{Z}^{m+c} \rightarrow \mathbb{L}_{A'}^\vee$, and a closed embedding $\overline{\varrho}' : \mathcal{KM} = (\mathbb{C}^*)^r \hookrightarrow W^*$. We will need to consider a slight twist of this morphism, which is given by composing it with the involution of $\iota : W^* \rightarrow W^*$ given by $\iota(\lambda_i) := (-1)^{\varepsilon(i)} \lambda_i$ with $\varepsilon(i) = 0$ for $i = 1, \dots, m$ and $\varepsilon(i) = 1$ for $i = m+1, \dots, m+c$. Denote by $\overline{\varrho}^* : \mathcal{KM} = (\mathbb{C}^*)^r \hookrightarrow W^*$ the composition $\iota \circ \overline{\varrho}'$. We will write $\overline{\varrho}$ for the composition with the canonical open embedding $W^* \hookrightarrow W = \text{Spec } \mathbb{C}[\lambda_1, \dots, \lambda_{m+c}]$.

We will further restrict our objects of study to that part of the complexified Kähler moduli space which maps to the set of good parameters in $W = \mathbb{C}^{m+c}$ as discussed in subsection 3.2. Hence we put $\mathcal{KM}^\circ := (\overline{\varrho})^{-1}(W^\circ) \subset \mathcal{KM}$, and we write $\varrho^* : \mathcal{KM}^\circ \hookrightarrow W^*$ and $\varrho : \mathcal{KM}^\circ \hookrightarrow W$ for the composition of $\overline{\varrho}^*$ with the inclusion $j : W^* \rightarrow W$.

We can now define the main object of study of this paper. We are going to use the constructions of the subsections 2.4 and 3.2, in particular, the diagrams (19), (23) and (33). We consider the composed morphism $\alpha \circ \beta : \mathcal{Z}_X \rightarrow \mathbb{C}_{\lambda_0} \times \mathcal{KM}$ as defined by diagram (23). Let $\mathcal{Z}_X^\circ := (\alpha \circ \beta)^{-1}(\mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ) \subset \mathcal{Z}_X$ be the subspace which is parameterized by the good parameter locus \mathcal{KM}° inside \mathcal{KM} .

For future reference, let us collect the relevant morphisms once again in a diagram, in which the spaces \mathcal{Z}° , $\mathcal{Z}_{X^{\text{aff}}}^\circ$, \mathcal{Z}_X° and \mathcal{G}° are defined by the requirement that all squares are cartesian. For simplicity of the notation, we denote by α, β, γ_1 and γ_2 also the corresponding restrictions above $\mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ$.

$$\begin{array}{ccccccc}
S & \xleftarrow{\pi_1^S} & \Gamma \cong S \times W & \xleftarrow{\quad} & \Gamma^* \cong S \times W^* & \xleftarrow{\quad} & \mathcal{G} \cong S \times \mathcal{KM} & \xleftarrow{\quad} & \mathcal{G}^\circ \cong S \times \mathcal{KM}^\circ \\
j_2 \downarrow & & \theta_2 \downarrow & & \zeta_2 \downarrow & & \gamma_2 \downarrow & & \gamma_2 \downarrow \\
X^{\text{aff}} & \xleftarrow{\quad} & Z_{X^{\text{aff}}} \cong X^{\text{aff}} \times V & \xleftarrow{\quad} & Z_{X^{\text{aff}}}^* \cong X^{\text{aff}} \times V^* & \xleftarrow{\quad} & Z_{X^{\text{aff}}} \cong X^{\text{aff}} \times \mathcal{KM} & \xleftarrow{\quad} & Z_{X^{\text{aff}}}^\circ \cong X^{\text{aff}} \times \mathcal{KM}^\circ \\
j_1 \downarrow & & \theta_1 \downarrow & & \zeta_1 \downarrow & & \gamma_1 \downarrow & & \gamma_1 \downarrow \\
X & \xleftarrow{\quad} & Z_X & \xleftarrow{\quad} & Z_X^* & \xleftarrow{\quad} & Z_X & \xleftarrow{\quad} & Z_X^\circ \\
i \downarrow & & \eta \downarrow & & \varepsilon \downarrow & & \beta \downarrow & & \beta \downarrow \\
\mathbb{P}(V') & \xleftarrow{\pi_1^Z} & Z & \xleftarrow{\quad} & Z^* & \xleftarrow{\quad} & \mathcal{Z} & \xleftarrow{\quad} & \mathcal{Z}^\circ \\
& & \pi_2^Z \downarrow & & \delta \downarrow & & \alpha \downarrow & & \alpha \downarrow \\
& & V & \xleftarrow{\quad} & V^* & \xleftarrow{\quad} & \mathbb{C}_{\lambda_0} \times \mathcal{KM} & \xleftarrow{\quad} & \mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ \\
& & & \xleftarrow{\text{id}_{\mathbb{C}_{\lambda_0}} \times \varrho} & & & & &
\end{array} \quad (59)$$

Definition 6.3. The *non-affine Landau-Ginzburg model* associated to $(X_\Sigma, \mathcal{L}_1, \dots, \mathcal{L}_c)$ is the morphism

$$\Pi : \mathcal{Z}_X^\circ \longrightarrow \mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ,$$

which is by definition the restriction of the universal family of hyperplane sections of X , i.e. of the morphism $\pi_2^\circ \circ \eta : Z_X \rightarrow V$ to the parameter space \mathcal{KM}° . We recall once again that X is defined as the closure of the embedding $g : S \rightarrow \mathbb{P}(V')$ sending (y_1, \dots, y_{n+c}) to $(1 : \underline{y}^{a'_1} : \dots : \underline{y}^{a'_{m+c}})$ where \underline{a}'_i are the columns of the matrix A' from Definition 4.6.

We also consider the restrictions $\pi_1 = \alpha \circ \beta \circ \gamma_1 : \mathcal{Z}_{X^{\text{aff}}}^\circ \cong X^{\text{aff}} \times \mathcal{KM}^\circ \rightarrow \mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ$ resp. $\pi_2 = \alpha \circ \beta \circ \gamma_1 \circ \gamma_2 : \mathcal{G}^\circ \cong S \times \mathcal{KM}^\circ \rightarrow \mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ$. These are nothing but the family of Laurent polynomials

$$(\underline{y}, \underline{q}) \longmapsto \left(- \sum_{i=1}^m \underline{q}^{m_i} \cdot \underline{y}^{a'_i} + \sum_{i=m+1}^{m+c} \underline{q}^{m_i} \cdot \underline{y}^{a'_i}, \underline{q} \right),$$

where the monomial $\underline{y}^{a'_i}$ is seen as an element of $\mathcal{O}_{X^{\text{aff}}}$ in the first case and as an element of \mathcal{O}_S in the second case. Here \underline{m}_i is the i 'th column of the matrix $M \in \text{Mat}(r \times (m+c), \mathbb{Z})$ from above. Notice that the first component of π has been split in two sums with opposite signs of each summand due to the action of the involution ι entering in the definition of the morphism $g : \mathcal{KM}^\circ \hookrightarrow W$. Both morphisms π_1 and π_2 are called the **affine Landau-Ginzburg model** of $(X_\Sigma, \mathcal{L}_1, \dots, \mathcal{L}_c)$.

As we will see later, the affine Landau-Ginzburg model is related to the twisted quantum \mathcal{D} -module $\text{QDM}(X_\Sigma, \mathcal{E})$ whereas the reduced quantum \mathcal{D} -module $\overline{\text{QDM}}(X_\Sigma, \mathcal{E})$ can be obtained from the non-affine Landau-Ginzburg model $\Pi : \mathcal{Z}_X^\circ \rightarrow \mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ$. The next results are parallel to [RS10, corollary 3.3. and corollary 3.4]. They show that the calculation of the Gauß-Manin system resp. the intersection cohomology \mathcal{D} -module from section 2 can be used to describe the corresponding objects for the morphism Π .

Lemma 6.4. Consider, as in subsection 3.1, the localized partial Fourier-Laplace transformation, this time with base \mathcal{KM}° , that is, let $j_\tau : \mathbb{C}_\tau^* \times \mathcal{KM}^\circ \hookrightarrow \mathbb{C}_\tau \times \mathcal{KM}^\circ$, $j_z : \mathbb{C}_\tau^* \times \mathcal{KM}^\circ \hookrightarrow \mathbb{C}_z \times \mathcal{KM}^\circ$ and put $\text{FL}_{\mathcal{KM}^\circ}^{\text{loc}} := j_{z,+} j_\tau^+ \text{FL}_{\mathcal{KM}^\circ}$. Then we have

1.

$$\text{FL}_{\mathcal{KM}^\circ}^{\text{loc}} (\mathcal{H}^0 \pi_{2+} \mathcal{O}_{S \times \mathcal{KM}^\circ}) \cong (\text{id}_{\mathbb{C}_z} \times \varrho)^+ \widehat{\mathcal{M}}_{A'}^{(-c, 0, 0)}.$$

Similarly, the isomorphism

$$\text{FL}_{\mathcal{KM}^\circ}^{\text{loc}} (\mathcal{H}^0 \pi_{2+} \mathcal{O}_{S \times \mathcal{KM}^\circ}) \cong (\text{id}_{\mathbb{C}_z} \times \varrho^*)^+ {}^* \widehat{\mathcal{N}}_{A'}^{(0, 0, 0)}$$

holds. Notice that the embeddings $(\text{id}_{\mathbb{C}_z} \times \varrho)$ resp. $(\text{id}_{\mathbb{C}_z} \times \varrho^*)$ are obviously non-characteristic for both of the modules $\widehat{\mathcal{M}}_{A'}^{(-c, 0, 0)}$ and ${}^* \widehat{\mathcal{N}}_{A'}^{(0, 0, 0)}$ as their singular locus is contained in

$$(\{0, \infty\} \times \mathcal{KM}^\circ) \cup (\mathbb{P}_z^1 \times (W \setminus \mathcal{KM}^\circ)) \quad \text{resp.} \quad (\{0, \infty\} \times \mathcal{KM}^\circ) \cup (\mathbb{P}_z^1 \times (W^* \setminus \mathcal{KM}^\circ)).$$

Hence, the complexes $(\text{id}_{\mathbb{C}_z} \times \varrho)^+ \widehat{\mathcal{M}}_{A'}^{(-c, 0, 0)}$ and $(\text{id}_{\mathbb{C}_z} \times \varrho^*)^+ {}^* \widehat{\mathcal{N}}_{A'}^{(0, 0, 0)}$ have cohomology only in degree zero.

2. Let $\widetilde{F} : X^{\text{aff}} \times \mathcal{KM}^\circ \rightarrow \mathbb{C}_{\lambda_0}$ be the first component of the morphism π_1 , then we have the following isomorphism of $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ -modules

$$H^{n+c}(\Omega_{X^{\text{aff}} \times \mathcal{KM}^\circ / \mathcal{KM}^\circ}(\log D)[z], zd - d\widetilde{F}) \cong (\text{id}_{\mathbb{C}_z} \times \varrho^*)^* \left({}^* \widehat{\mathcal{M}}_{A'}^{(-c, 0, 0)} \right). \quad (60)$$

3. Denote by $(-)'$ the duality functor in the category of locally free $\mathcal{O}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ -modules with meromorphic connection with poles along $\{0\} \times \mathcal{KM}^\circ$, that is, if (\mathcal{F}, ∇) is an object of this category, we put $(\mathcal{F}, \nabla)' := (\text{Hom}_{\mathcal{O}_{\mathbb{C}_z \times \mathcal{KM}^\circ}}(\mathcal{F}, \mathcal{O}_{\mathbb{C}_z \times \mathcal{KM}^\circ}), \nabla')$, where ∇' is the dual connection. Notice that the $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ -modules from isomorphism (60) are actually objects of this category. Notice also that the duality functor in the category of $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ -modules (i.e., the functor $\text{Ext}_{\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}}^{r+1}(-, \mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ})$) restricts to $(-)'$ on the subcategory described above (this follows from [DS03, Lemma A.12]).

There is an isomorphism of $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ -modules

$$\left(H^{n+c}(\Omega_{X^{\text{aff}} \times \mathcal{KM}^\circ / \mathcal{KM}^\circ}(\log D)[z], zd - d\widetilde{F}) \right)' \xrightarrow{\cong} (\text{id}_{\mathbb{C}_z} \times \varrho^*)^* \left({}^* \widehat{\mathcal{N}}_{A'}^{(0, 0, 0)} \right).$$

Proof. 1. The proof of the first isomorphism is the same as [RS10, corollary 3.3]: Consider the cartesian diagram (which is part of the diagram (59))

$$\begin{array}{ccc} \mathcal{G}^\circ \cong S \times \mathcal{KM}^\circ & \longrightarrow & \Gamma \cong S \times W \\ \downarrow \pi_2 & & \downarrow \varphi \\ \mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ & \xrightarrow{\text{id}_{\mathbb{C}_{\lambda_0}} \times \varrho} & V = \mathbb{C}_{\lambda_0} \times W \end{array} \quad (61)$$

then the base change property (Theorem 2.1) and the commutation of FL^{loc} with inverse images shows that

$$\text{FL}_{\mathcal{KM}^\circ}^{\text{loc}}(\mathcal{H}^0 \pi_{2+} \mathcal{O}_{S \times \mathcal{KM}^\circ}) \cong (\text{id}_{\mathbb{C}_z} \times \varrho)^+ \mathcal{G}^+,$$

where \mathcal{G}^+ is the $\mathcal{D}_{\mathbb{C}_{\lambda_0} \times W}$ -module introduced in subsection 3.1, and then one concludes using Proposition 3.3.

Concerning the second isomorphism, we use base change (with respect to the morphism $\text{id}_{\mathbb{C}_{\lambda_0}} \times \varrho$ in diagram (61)) for proper direct images and exceptional inverse images. However, the latter ones equal ordinary inverse images if the horizontal morphisms in the above diagram are non-characteristic for the modules in question. This is the case by Proposition 2.22, 2., so that we obtain

$$\text{FL}_{\mathcal{KM}^\circ}^{\text{loc}}(\mathcal{H}^0 \pi_{2+} \mathcal{O}_{S \times \mathcal{KM}^\circ}) \cong (\text{id}_{\mathbb{C}_z} \times \varrho)^+ \text{FL}_W^{\text{loc}}(\mathcal{H}^0 \varphi_{B,\dagger} \mathcal{O}_{S \times W}) = (\text{id}_{\mathbb{C}_z} \times \varrho)^+ \mathcal{G}^\dagger.$$

The second part of corollary 5.5 (and the second part of Proposition 3.3) tells us that $\mathcal{G}^\dagger \cong \widehat{\mathcal{M}}_{A'}^{-(c,0,1)}$. However, the isomorphism $\Psi : {}^* \widehat{\mathcal{N}}_{A'}^{(0,0,0)} \longrightarrow {}^* \widehat{\mathcal{M}}_{A'}^{-(2c,0,1)}$ given by right multiplication with $z^c \cdot \lambda_{m+1} \cdot \dots \cdot \lambda_{m+c}$ (see equation (52)) shows that

$$(\text{id}_{\mathbb{C}_z} \times \varrho)^+ \widehat{\mathcal{M}}_{A'}^{-(2c,0,1)} \cong (\text{id}_{\mathbb{C}_z} \times \varrho^*)^+ {}^* \widehat{\mathcal{N}}_{A'}^{(0,0,0)}$$

so that finally we arrive at the desired equality

$$\text{FL}_{\mathcal{KM}^\circ}^{\text{loc}}(\mathcal{H}^0 \pi_{2+} \mathcal{O}_{S \times \mathcal{KM}^\circ}) \cong (\text{id}_{\mathbb{C}_z} \times \varrho^*)^+ {}^* \widehat{\mathcal{N}}_{A'}^{(0,0,0)}.$$

2. In order to show the statement, notice that by definition $H^{n+c}(\Omega_{X^{\text{aff}} \times W^*/W^*}^\bullet(\log D)[z], zd - d\widetilde{F})$ is the cokernel of

$$\Omega_{X^{\text{aff}} \times W^*/W^*}^{n+c-1}(\log D)[z] \xrightarrow{zd - d\widetilde{F}} \Omega_{X^{\text{aff}} \times W^*/W^*}^{n+c}(\log D)[z],$$

that is, the cokernel of an $\mathcal{O}_{\mathbb{C}_z \times W^*}$ -linear morphism between free (though not coherent) $\mathcal{O}_{\mathbb{C}_z \times W^*}$ -modules. Hence tensoring with $\mathcal{O}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ yields the exact sequence

$$\begin{aligned} & \Omega_{X^{\text{aff}} \times \mathcal{KM}^\circ / \mathcal{KM}^\circ}^{n+c-1}(\log D)[z] \xrightarrow{zd - d\widetilde{F}} \Omega_{X^{\text{aff}} \times \mathcal{KM}^\circ / \mathcal{KM}^\circ}^{n+c}(\log D)[z] \longrightarrow \\ & \mathcal{O}_{\mathbb{C}_z \times \mathcal{KM}^\circ} \otimes_{\mathcal{O}_{\mathbb{C}_z \times W^*}} H^{n+c}(\Omega_{X^{\text{aff}} \times W^*/W^*}^\bullet(\log D)[z], zd - d\widetilde{F}) \longrightarrow 0 \end{aligned}$$

from which we conclude that

$$\begin{aligned} & H^{n+c}(\Omega_{X^{\text{aff}} \times \mathcal{KM}^\circ / \mathcal{KM}^\circ}^\bullet(\log D)[z], zd - d\widetilde{F}) = \\ & \mathcal{O}_{\mathbb{C}_z \times \mathcal{KM}^\circ} \otimes_{\mathcal{O}_{\mathbb{C}_z \times W^*}} H^{n+c}(\Omega_{X^{\text{aff}} \times W^*/W^*}^\bullet(\log D)[z], zd - d\widetilde{F}). \end{aligned}$$

We know by Proposition 3.20 that

$$H^{n+c}(\Omega_{X^{\text{aff}} \times W^*/W^*}^\bullet(\log D)[z], zd - d\widetilde{F}) \cong {}^0 \widehat{\mathcal{M}}_{A'}^{(-c,0,0)}.$$

Hence the desired statement, i.e., Formula (60) follows.

3. Consider the filtration on $\mathcal{D}_{\mathbb{C}_z \times W}$ resp. on $\mathcal{D}_{\mathbb{C}_z \times W^*}$ which extends the order filtration on \mathcal{D}_W (resp. on \mathcal{D}_{W^*}) and for which z has degree -1 and ∂_z has degree 2 . Denote by G_\bullet the induced filtrations on the modules ${}^*\widehat{\mathcal{N}}_{A'}^{(0,0,0)}$ and $\widehat{\mathcal{M}}_{A'}^{(-c,0,0)}$ resp. on ${}^*\widehat{\mathcal{M}}_{A'}^{(-c,0,0)}$, in particular, we have $G_0({}^*\widehat{\mathcal{N}}_{A'}^{(0,0,0)}) = {}_0\widehat{\mathcal{N}}_{A'}^{(0,0,0)}$ and $G_0(\widehat{\mathcal{M}}_{A'}^{(-c,0,0)}) = {}_0\widehat{\mathcal{M}}_{A'}^{(-c,0,0)}$ resp. $G_0({}^*\widehat{\mathcal{M}}_{A'}^{(-c,0,0)}) = {}_0{}^*\widehat{\mathcal{M}}_{A'}^{(-c,0,0)}$.

Similar to the proof of [RS10, Proposition 2.18, 3.], we consider the saturation of the filtration F_\bullet on $\mathcal{M}_{A''}^\beta$ by $\partial_{\lambda_0}^{-1}$. More precisely, we first notice that Lemma 3.2 can be reformulated by saying that for any $\beta' = (\beta'_0, \beta'_1, \dots, \beta'_{n+c}) \in \mathbb{Z}^{1+n+c}$, we have

$$\widehat{\mathcal{M}}_{A'}^\beta = \text{FL}_W(\mathcal{M}_{A''}^{\beta'}[\partial_{\lambda_0}^{-1}]),$$

where $\beta_0 = \beta'_0 + 1$ and $\beta_i = \beta'_i$ for $i = 1, \dots, n+c$ and where we write $\mathcal{M}_{A''}^{\beta'}[\partial_{\lambda_0}^{-1}] := \mathcal{D}_V[\partial_{\lambda_0}^{-1}] \otimes_{\mathcal{D}_V} \mathcal{M}_{A''}^{\beta'}$. Now we consider the natural localization morphism $\widehat{\text{loc}} : \mathcal{M}_{A''}^{\beta'} \rightarrow \mathcal{M}_{A''}^{\beta'}[\partial_{\lambda_0}^{-1}]$ and we put

$$F_k \mathcal{M}_{A''}^{\beta'}[\partial_{\lambda_0}^{-1}] := \sum_{j \geq 0} \partial_{\lambda_0}^{-j} F_{k+j} \mathcal{M}_{A''}^{\beta'}.$$

As we have

$$F_k \mathcal{M}_{A''}^{\beta'}[\partial_{\lambda_0}^{-1}] = \text{im}(\partial_{\lambda_0}^k \mathbb{C}[\lambda_0, \lambda_1, \dots, \lambda_{m+c}] \langle \partial_{\lambda_0}^{-1}, \partial_{\lambda_0}^{-1} \partial_{\lambda_1}, \dots, \partial_{\lambda_0}^{-1} \partial_{\lambda_{m+c}} \rangle) \text{ in } \mathcal{M}_{A''}^{\beta'}[\partial_{\lambda_0}^{-1}],$$

the filtration induced by $F_k \mathcal{M}_{A''}^{\beta'}[\partial_{\lambda_0}^{-1}]$ on $\widehat{\mathcal{M}}_{A'}^\beta$ is precisely $G_k \widehat{\mathcal{M}}_{A'}^\beta$. From [Sai89, formula 2.7.5] we conclude that

$$(G_l \widehat{\mathcal{M}}_{A'}^{-(c,0,1)})' = \text{Hom}_{\mathcal{O}_{\mathbb{C}_z \times W}}(G_l \widehat{\mathcal{M}}_{A'}^{-(c,0,1)}, \mathcal{O}_{\mathbb{C}_z \times W}) \stackrel{!}{=} G_{l+(m+c+2)}^{\mathbb{D}} \widehat{\mathcal{M}}_{A'}^{(1,0,0)},$$

where $G^{\mathbb{D}} \widehat{\mathcal{M}}_{A'}^{(1,0,0)}$ is the filtration induced by the saturation of the filtration on $\mathcal{M}_{A''}^{(0,0,0)}$ dual to the order filtration F_\bullet on $\mathcal{M}_{A''}^{-(c+1,0,1)}$. By Theorem 5.4, 2. and by restriction to $\mathbb{C}_z \times W^*$ we obtain

$$G_\bullet^{\mathbb{D}} {}^*\widehat{\mathcal{M}}_{A'}^{(1,0,0)} = G_{\bullet+n-(m+c+1)} {}^*\widehat{\mathcal{M}}_{A'}^{(1,0,0)},$$

so that

$$(G_l {}^*\widehat{\mathcal{M}}_{A'}^{-(c,0,1)})' = G_{l+n+1} {}^*\widehat{\mathcal{M}}_{A'}^{(1,0,0)}.$$

Multiplying this equation by z^c gives

$$(G_k {}^*\widehat{\mathcal{M}}_{A'}^{-(2c,0,1)})' = G_{k+n+1} {}^*\widehat{\mathcal{M}}_{A'}^{-(c-1,0,0)},$$

where $k = l - c$. Moreover, we evidently have

$$\cdot z : G_{k+n+1} {}^*\widehat{\mathcal{M}}_{A'}^{-(c-1,0,0)} \xrightarrow{\cong} G_{k+n} {}^*\widehat{\mathcal{M}}_{A'}^{-(c,0,0)},$$

so that

$$(G_\bullet {}^*\widehat{\mathcal{M}}_{A'}^{-(2c,0,1)})' \cong G_{\bullet+n} {}^*\widehat{\mathcal{M}}_{A'}^{-(c,0,0)}.$$

The isomorphism Ψ from Formula (52) satisfies

$$\Psi : G_k {}^*\widehat{\mathcal{N}}_{A'}^{(0,0,0)} \xrightarrow{\cong} G_{k-c} {}^*\widehat{\mathcal{M}}_{A'}^{-(2c,0,1)}.$$

In conclusion (i.e., putting $k = 0$), we obtain

$$\left({}_0\widehat{\mathcal{N}}_{A'}^{(0,0,0)} \right)' \cong z^{c-n} \cdot {}_0{}^*\widehat{\mathcal{M}}_{A'}^{-(c,0,0)} \subset {}^*\widehat{\mathcal{M}}_{A'}^{-(c,0,0)}$$

and then the statement follows from part 2. from above, as multiplication with z is invertible and as the inverse image under $\text{id}_{\mathbb{C}_z} \times \varrho^*$ commutes with the functor $(-)'$. \square

Now we can construct a $\mathcal{D}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ -module from the non-affine Landau-Ginzburg model $\Pi : \mathcal{Z}_X^\circ \rightarrow \mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ$ that will ultimately give us the reduced quantum \mathcal{D} -module. It will consist in a minimal extension of the local system of intersection cohomologies of the fibres of Π .

Proposition 6.5. 1. Consider the local system \mathcal{L} and \mathcal{O}_V -free module \mathcal{C}_0 from Proposition 2.13. Then

$$\mathcal{H}^0 \alpha_+ \mathcal{M}^{IC}(\mathcal{Z}_X^\circ) \cong (\mathrm{id}_{\mathbb{C}_{\lambda_0}} \times \varrho)^+ (\mathcal{M}^{IC}(X^\circ, \mathcal{L}) \oplus \mathcal{C}_0).$$

Recall that $\mathcal{M}^{IC}(\mathcal{Z}_X^\circ)$ is the intersection cohomology \mathcal{D} -module of \mathcal{Z}_X° , that is, the unique regular singular $\mathcal{D}_{\mathcal{Z}^\circ}$ -module supported on \mathcal{Z}_X° which corresponds to the intermediate extension of the constant sheaf on the smooth part of \mathcal{Z}_X° .

Using the Riemann-Hilbert correspondence, the above isomorphism can be expressed in terms of the morphism Π as

$${}^p\mathcal{H}^0 R\Pi_* IC(\mathcal{Z}_X^\circ) \cong (\mathrm{id}_{\mathbb{C}_{\lambda_0}} \times \varrho)^{-1} ((j_{X^\circ})_! IC(X^\circ, \mathcal{L}) \oplus \mathcal{C}_0^\nabla),$$

where $j_{X^\circ} : X_0 \hookrightarrow V$ is the canonical closed embedding, where \mathcal{C}_0^∇ is the local system corresponding to \mathcal{C}_0 and where ${}^p\mathcal{H}$ denotes the perverse cohomology functor.

2. We have isomorphisms of $\mathcal{D}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ -modules

$$\mathrm{FL}_{\mathcal{KM}^\circ}^{loc} (\mathcal{H}^0 \alpha_+ \mathcal{M}^{IC}(\mathcal{Z}_X^\circ)) \cong (\mathrm{id}_{\mathbb{C}_z} \times \varrho)^+ \widehat{\mathcal{M}}^{IC}(X^\circ, \mathcal{L}) \cong (\mathrm{id}_{\mathbb{C}_z} \times \varrho^*)^+ \mathrm{im}(\tilde{\phi}),$$

where $\tilde{\phi} : {}^*\widehat{\mathcal{N}}_{A'}^{(0,0,0)} \rightarrow {}^*\widehat{\mathcal{M}}_{A'}^{(-c,0,0)}$ is the morphism introduced in Definition 6.1.

Proof. 1. As the inclusion $\mathcal{Z}^\circ \hookrightarrow \mathcal{Z}$ is open and hence non-characteristic for any $\mathcal{D}_{\mathcal{Z}}$ -module, the assertion to be shown follows from Proposition 2.22 (more precisely, from Formula (24)) and Proposition 2.13.

2. The first isomorphism is a direct consequence of the last point, using again the commutation of FL^{loc} with the inverse image and the fact that the \mathcal{O}_V -free module \mathcal{C}_0 is killed by FL_W^{loc} . The second isomorphism follows from equation (53). \square

For future use, we give names to the \mathcal{D} -modules on the Kähler moduli space considered above. We also define natural lattices inside them.

Definition 6.6. Define the following $\mathcal{D}_{\mathbb{C}_z \times \mathcal{KM}}$ -modules:

$$\mathcal{QM}_{A'} := (\mathrm{id}_{\mathbb{C}_z} \times \bar{\varrho}^*)^+ \left({}^*\widehat{\mathcal{N}}_{A'}^{(0,0,0)} \right) \quad \text{and} \quad \mathcal{QM}_{A'}^{IC} := (\mathrm{id}_{\mathbb{C}_z} \times \bar{\varrho}^*)^+ \left(\mathrm{im}(\tilde{\phi}) \right).$$

Define moreover

$${}_0\mathcal{QM}_{A'} := (\mathrm{id}_{\mathbb{C}_z} \times \bar{\varrho}^*)^* \left({}^*\widehat{\mathcal{N}}_{A'}^{(0,0,0)} \right) \quad \text{and} \quad {}_0\mathcal{QM}_{A'}^{IC} := (\mathrm{id}_{\mathbb{C}_z} \times \bar{\varrho}^*)^* \left(\tilde{\phi} \left({}^*\widehat{\mathcal{N}}_{A'}^{(0,0,0)} \right) \right),$$

where here the functor $(\mathrm{id}_{\mathbb{C}_z} \times \bar{\varrho}^*)^*$ is the inverse image in the category of holomorphic vector bundles on $\mathbb{C}_z \times \mathcal{KM}$ with meromorphic connection (meromorphic along $\{0\} \times \mathcal{KM}$).

We proceed by comparing the objects $\mathcal{QM}_{A'}$ and $\mathcal{QM}_{A'}^{IC}$ just introduced to the twisted and the reduced quantum \mathcal{D} -module from section 4. For the readers convenience, let us recall one of the main results from [MM11] which concerns the toric description of the twisted resp. reduced quantum \mathcal{D} -modules.

Theorem 6.7 ([MM11, Theorem 5.10]). Let X_Σ as before, and suppose that $\mathcal{L}_1, \dots, \mathcal{L}_c$ are ample line bundles on X_Σ such that $-K_{X_\Sigma} - \sum_{j=1}^c \mathcal{L}_j$ is nef. Put again $\mathcal{E} := \bigoplus_{j=1}^c \mathcal{L}_j$. For any $\mathcal{L} \in \mathrm{Pic}(X_\Sigma)$ with $c_1(\mathcal{L}) = \sum_{a=1}^r d_a p_a \in \mathbb{L}_{A'}^\vee$, we put $\widehat{\mathcal{L}} = \sum_{a=1}^r z d_a q_a \partial_{q_a} \in R_{\mathbb{C}_z \times \mathcal{KM}}$. Define the left ideal J of $R_{\mathbb{C}_z \times \mathcal{KM}}$ by

$$J := R_{\mathbb{C}_z \times \mathcal{KM}}(Q_{\underline{l}})_{\underline{l} \in \mathbb{L}_{A'}} + R_{\mathbb{C}_z \times \mathcal{KM}} \cdot \widehat{E},$$

where

$$Q_{\mathcal{L}} := \prod_{i \in \{1, \dots, m\}: l_i > 0} \prod_{\nu=0}^{l_i-1} (\widehat{\mathcal{D}}_i - \nu z) \prod_{j \in \{1, \dots, c\}: l_{m+j} > 0} \prod_{\nu=1}^{l_{m+j}} (\widehat{\mathcal{L}}_j + \nu z) \\ - \frac{q^l}{z} \prod_{i \in \{1, \dots, m\}: l_i < 0} \prod_{\nu=0}^{-l_i-1} (\widehat{\mathcal{D}}_i - \nu z) \prod_{j \in \{1, \dots, c\}: l_{m+j} < 0} \prod_{\nu=1}^{-l_{m+j}} (\widehat{\mathcal{L}}_j + \nu z),$$

$$\widehat{E} := z^2 \partial_z - \widehat{K}_{\mathbb{V}(\mathcal{E}^\vee)}.$$

Here we write $\mathcal{D}_i \in \text{Pic}(X_\Sigma)$ for a line bundle associated to the torus invariant divisor D_i , where $i = 1, \dots, m$. Notice that the ideal J was called \mathbb{G} in [MM11, Definition 4.3].

Moreover, let Quot be the left ideal in $R_{\mathbb{C}_z \times \mathcal{KM}}$ generated by the following set

$$G := \{P \in R_{\mathbb{C}_z \times \mathcal{KM}} \mid \widehat{c}_{\text{top}} \cdot P \in J\},$$

where $\widehat{c}_{\text{top}} := \prod_{j=1}^c \widehat{\mathcal{L}}_j$. We define $\mathcal{P} := R_{\mathbb{C}_z \times \mathcal{KM}}/J$ resp. $\mathcal{P}^{\text{res}} := R_{\mathbb{C}_z \times \mathcal{KM}}/\text{Quot}$ and denote by $\mathcal{P} = \mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}}/\mathcal{J}$ resp. $\mathcal{P}^{\text{res}} = \mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}}/\text{Quot}$ the corresponding $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}}$ -modules. Notice that we have $\mathcal{J} \subset \text{Quot}$, hence there is a canonical surjection $\mathcal{P} \twoheadrightarrow \mathcal{P}^{\text{res}}$.

Put $B_\varepsilon^* := \{q \in (\mathbb{C}^*)^r \mid 0 < |q| < \varepsilon\} \subset \mathcal{KM}^\circ$, then there is some ε such that the following diagram is commutative and the horizontal morphisms are isomorphisms of $\mathcal{R}_{\mathbb{C}_z \times B_\varepsilon^*}$ -modules.

$$\begin{array}{ccc} \mathcal{P}|_{\mathbb{C}_z \times B_\varepsilon^*} & \xrightarrow{\cong} & (\text{id}_{\mathbb{C}_z} \times \text{Mir})^* (\text{QDM}(X_\Sigma, \mathcal{E}))|_{\mathbb{C}_z \times B_\varepsilon^*} \\ \downarrow & & \downarrow \bar{\pi} \\ \mathcal{P}^{\text{res}}|_{\mathbb{C}_z \times B_\varepsilon^*} & \xrightarrow{\cong} & (\text{id}_{\mathbb{C}_z} \times \text{Mir})^* (\overline{\text{QDM}}(X_\Sigma, \mathcal{E}))|_{\mathbb{C}_z \times B_\varepsilon^*} \end{array}$$

Here Mir is the **mirror map**, as described, e.g., in [MM11, Theorem 5.6].

We now define another quotient \mathcal{Q}^{res} of \mathcal{P} which is better suited to our approach and which turns out to be isomorphic to \mathcal{P}^{res} resp. to $(\text{id}_{\mathbb{C}_z} \times \text{Mir})^* (\overline{\text{QDM}}(X_\Sigma, \mathcal{E}))$ in some neighborhood of $q = 0$.

Definition 6.8. Let K be the following ideal in $R_{\mathbb{C}_z \times \mathcal{KM}}$:

$$K := \{P \in R_{\mathbb{C}_z \times \mathcal{KM}} \mid \exists p \in \mathbb{Z}, k \in \mathbb{N} \text{ such that } \prod_{i=0}^k \widehat{c}_{\text{top}}^{p+i} P \in J\},$$

where $\widehat{c}_{\text{top}}^i := \prod_{j=1}^c (\widehat{\mathcal{L}}_j + i)$. Define

$$\mathcal{Q}^{\text{res}} := R_{\mathbb{C}_z \times \mathcal{KM}}/K$$

and denote by \mathcal{Q}^{res} be the corresponding $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}}$ -module.

Proposition 6.9. Using the notations from above, we have the following isomorphisms:

$$\mathcal{P}^{\text{res}}|_{\mathbb{C}_z \times B_\varepsilon^*} \simeq \mathcal{Q}^{\text{res}}|_{\mathbb{C}_z \times B_\varepsilon^*} \simeq (\overline{\text{QDM}}(X_\Sigma, \mathcal{E}))|_{\mathbb{C}_z \times B_\varepsilon^*}.$$

Proof. First notice that we have a surjective morphism $\mathcal{P}^{\text{res}} \twoheadrightarrow \mathcal{Q}^{\text{res}}$ because the generating set G of Quot is contained in the ideal K . If we can construct a well-defined morphism

$$\mathcal{Q}^{\text{res}}|_{\mathbb{C}_z \times B_\varepsilon^*} \rightarrow (\overline{\text{QDM}}(X_\Sigma, \mathcal{E}))|_{\mathbb{C}_z \times B_\varepsilon^*} \quad (62)$$

such that the following diagram

$$\begin{array}{ccc} \mathcal{P}^{\text{res}}|_{\mathbb{C}_z \times B_\varepsilon^*} & \xrightarrow{\twoheadrightarrow} & \mathcal{Q}^{\text{res}}|_{\mathbb{C}_z \times B_\varepsilon^*} \\ \downarrow \simeq & \swarrow & \\ (\overline{\text{QDM}}(X_\Sigma, \mathcal{E}))|_{\mathbb{C}_z \times B_\varepsilon^*} & & \end{array}$$

commutes, the proposition follows. In order to construct the morphism (62) we recapitulate the construction from [MM11] of the morphisms $\mathcal{P}_{|\mathbb{C}_z \times B_\varepsilon^*} \rightarrow (\text{QDM}(X_\Sigma, \mathcal{E}))_{|\mathbb{C}_z \times B_\varepsilon^*}$ resp. $\mathcal{P}_{|\mathbb{C}_z \times B_\varepsilon^*}^{res} \rightarrow (\overline{\text{QDM}}(X_\Sigma, \mathcal{E}))_{|\mathbb{C}_z \times B_\varepsilon^*}$. They define a multivalued section L^{tw} in $\text{End}(\text{QDM}(X_\Sigma, \mathcal{E}))$ such that $L^{tw} z^{-\mu} z^{c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee)}$ is a fundamental solution of $\text{QDM}(X_\Sigma, \mathcal{E})$ (cf. [MM11, Proposition 2.17]) and a multi-valued section J^{tw} with the property that

$$J^{tw} := (L^{tw})^{-1} 1 \quad \text{in} \quad \overline{\text{QDM}}(X_\Sigma, \mathcal{E}).$$

We need yet another cohomological multi-valued section

$$I := q^{T/z} \sum_{d \in H_2(X, \mathbb{Z})} q^d A_d(z),$$

where

$$A_d(z) := \prod_{i=1}^c \frac{\prod_{m=-\infty}^{d_{L_i}} ([L_i] + mz)}{\prod_{m=-\infty}^0 ([L_i] + mz)} \prod_{\theta \in \Sigma(1)} \frac{\prod_{m=-\infty}^0 ([D_\theta] + mz)}{\prod_{m=-\infty}^{d_\theta} ([D_\theta] + mz)},$$

$$q^{T/z} := e^{\frac{1}{z} \sum_{a=1}^r T_A \log(q_a)},$$

$d_\theta := \int_d D_\theta$ and $d_{L_i} := \int_d c_1(\mathcal{L}_i)$ and which has asymptotic development $I = F(q)1 + O(z^{-1})$. The mirror theorem of Givental resp. Coates-Givental (see e.g. [MM11, Theorem 5.6]) says that

$$I(q, z) = F(q) \cdot J^{tw}(\text{Mir}(q), z).$$

Now one defines the following morphism

$$\mathcal{R}_{\mathbb{C}_z \times B_\varepsilon^*} \longrightarrow (id \times \text{Mir})^* (\text{QDM}(X_\Sigma, \mathcal{E}))_{|\mathbb{C}_z \times B_\varepsilon^*}, \quad (63)$$

$$P(z, q, zq\partial_q, z^2\partial_z) \mapsto L^{tw}(\text{Mir}(q), z) z^{-\mu} z^{c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee)} P(q, z, z\partial_q, z^2\partial_z) z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee)} z^\mu F(q) J^{tw}(\text{Mir}(q), z)$$

$$= L^{tw}(\text{Mir}(q), z) z^{-\mu} z^{c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee)} P(q, z, z\partial_q, z^2\partial_z) z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee)} z^\mu I(q, z)$$

whose surjectivity is shown in the proof of Theorem 5.10 in [MM11].

The morphism above descends to $\mathcal{P}_{|\mathbb{C}_z \times B_\varepsilon^*}$ by the fact that

$$P(q, z, zq\partial_q, z^2\partial_z) z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee)} z^\mu I = 0 \quad \text{for} \quad P \in \mathcal{J}.$$

If one composes the morphism (63) with the quotient morphism $\bar{\pi}$, then this descends to a morphism

$$\mathcal{P}_{|\mathbb{C}_z \times B_\varepsilon^*}^{res} \rightarrow (\overline{\text{QDM}}(X_\Sigma, \mathcal{E}))_{|\mathbb{C}_z \times B_\varepsilon^*}, \quad (64)$$

which follows from

$$P(q, z, zq\partial_q, z^2\partial_z) z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee)} z^\mu I \in \ker(m_{c_{top}}) \quad \text{for} \quad P \in \text{Quot} \quad (65)$$

and the fact that L^{tw} preserves $\ker(m_{c_{top}})$ (cf. [MM11, Lemma 2.31]).

As explained above, the proposition will follow if the morphism (64) descends to $\mathcal{Q}_{|\mathbb{C}_z \times B_\varepsilon^*}^{res}$, i.e. we have to show that

$$P(q, z, zq\partial_q, z^2\partial_z) z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee)} z^\mu I \in \ker(m_{c_{top}}) \quad \text{for} \quad P \in \mathcal{K}. \quad (66)$$

We will adapt the proof of (65) from [MM11, Lemma 5.21] to our situation. First notice that

$$z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee)} z^\mu I = \sum_{d \in H_2(X, \mathbb{Z})} q^{T+d} z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee) - d_{\mathcal{T}_X \otimes \mathcal{E}^\vee}} A_d(1).$$

Now let $P(q, z, zq\partial_q, z^2\partial_z) \in \mathcal{K}$ and decompose it:

$$P(q, z, zq\partial_q, z^2\partial_z) = \sum_{\substack{d' \in H_2(X, \mathbb{Z}) \\ \text{finite}}} q^{d'} P_{d'}(z, z\partial_q, z^2\partial_z).$$

This gives

$$P(q, z, q\partial_q, z^2\partial_z)z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee)}z^\mu I = \sum_{d \in H_2(X, \mathbb{Z})} q^{T+d} z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee) - d_{\mathcal{T}_X \otimes \mathcal{E}^\vee}} B_d(z),$$

where

$$B_d(z) := \sum_{\substack{d' \in H_2(X, \mathbb{Z}) \\ \text{finite}}} P_{d'}(z, z(T+d), z(-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee) - d_{\mathcal{T}_X \otimes \mathcal{E}^\vee})) A_{d-d'}(1).$$

Similarly to loc. cit., the statement (66) will follow from the fact that $c_{top}B_d(z) = 0$ for all $d \in H_2(X, \mathbb{Z})$. Because $P \in K$, there exists $p \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that

$$\left(\prod_{i=0}^k \widehat{c}_{top}^{p+i} \right) P(q, z, zq\partial_q, z^2\partial_z)z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee)}z^\mu I = 0,$$

which gives

$$\sum_{d \in H_2(X, \mathbb{Z})} q^{T+d} z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee) - d_{\mathcal{T}_X \otimes \mathcal{E}^\vee}} \left(\prod_{i=0}^k \prod_{j=1}^c z([L_j] + d_{L_j} + p + i) \right) B_d(z) = 0.$$

Notice that the sum above is zero if and only if each summand is zero. For $(z, q) \in \mathbb{C}_z^* \times W$ the term $q^{T+d} z^{-c_1(\mathcal{T}_X \otimes \mathcal{E}^\vee) - d_{\mathcal{T}_X \otimes \mathcal{E}^\vee}}$ is invertible, so we deduce that

$$\left(\prod_{i=0}^k \prod_{j=1}^c ([L_j] + d_{L_j} + p + i) \right) B_d(z) = 0 \quad \forall d \in H_2(X, \mathbb{Z}).$$

Let $J_d := \{j \in \{1, \dots, c\} \mid \exists i \in \{0, \dots, k\} \text{ with } d_{L_j} + p + j = 0\}$ and notice that for every j there is at most one $i \in \{0, \dots, k\}$ such that $d_{L_j} + p + i = 0$. Because cup-product with $[L_j] + l$ is an automorphism of $H^{2*}(X, \mathbb{C})$ for every $l \neq 0$, we conclude that

$$\left(\prod_{j \in J_d} [L_j] \right) B_d(z) = 0 \quad \forall d \in H_2(X, \mathbb{Z}),$$

which in turn shows that $c_{top}B_d(z) = (\prod_{j=1}^c [L_j])B_d(z) = 0$ for all $d \in H_2(X, \mathbb{Z})$. \square

The next proposition compares the \mathcal{R} -modules from Theorem 6.7 and Definition 6.8 with ${}_0\mathcal{QM}_{A'}$ and ${}_0\mathcal{QM}_{A'}^{IC}$.

Proposition 6.10. *We have isomorphisms of $\mathcal{R}_{\mathbb{C}_z \times \mathcal{KM}^\circ}$ -modules*

$$\mathcal{P} \cong {}_0\mathcal{QM}_{A'} \quad \text{and} \quad \mathcal{Q}^{\text{res}} \cong {}_0\mathcal{QM}_{A'}^{IC}.$$

Proof. The first isomorphism follows from a similar argument as [RS10, Proposition 3.2], namely, the section

$$\begin{aligned} \bar{\varrho} : \mathcal{KM} &\hookrightarrow W^*, \\ (q_1, \dots, q_r) &\mapsto (\lambda_1 = \underline{q}^{m_1}, \dots, \lambda_m = \underline{q}^{m_m}, \lambda_{m+1} = -\underline{q}^{m_{m+1}}, \dots, \lambda_{m+c} = -\underline{q}^{m_{m+c}}) \end{aligned}$$

can be used to construct an isomorphism

$$\begin{aligned} \theta : F \times \mathcal{KM} &\longrightarrow W^*, \\ (f_1, \dots, f_{n+c}, q_1, \dots, q_r) &\mapsto (\underline{q}^{m_1} \underline{y}^{a_1}, \dots, \underline{q}^{m_m} \underline{y}^{a_m}, -\underline{q}^{m_{m+1}} \underline{y}^{a_{m+1}}, \dots, -\underline{q}^{m_{m+c}} \underline{y}^{a_{m+c}}) \end{aligned}$$

with inverse

$$\begin{aligned} \theta^{-1} : W^* &\longrightarrow F \times \mathcal{KM}, \\ (\lambda_1, \dots, \lambda_{m+c}) &\mapsto (f_j = (-1)^{\sum_{i=m+1}^{m+c} c_{ij}} \underline{\lambda}^{\varepsilon_j}, q_a = (-1)^{\sum_{i=m+1}^{m+c} l_{ia}} \underline{\lambda}^{l_a}), \end{aligned}$$

where $L = (L_a)$ resp. $M = (m_i)$ are the matrices which were introduced above Definition 6.3 and $C = (c_j)$ is a $(m+c) \times (n+c)$ -matrix such that the following equations are fulfilled (cf. Section 2.4):

$$M \cdot L = I_r, \quad B \cdot C = I_{n+c}, \quad B \cdot L = 0, \quad M \cdot C = 0, \quad C \cdot B + L \cdot M = I_{m+c}.$$

Under this coordinate change the module ${}^0\widehat{\mathcal{N}}_{A'}^{(0,0,0)}$ has the following presentation:

$$\mathcal{R}_{\mathbb{C}_z \times F \times \mathcal{KM}} / ((Q_l)_{l \in \mathbb{L}} + (\widehat{E}) + (\widehat{E}'_k)_{k=1, \dots, n+c})$$

with Q_l and \widehat{E} as in Definition 6.7 and $\widehat{E}'_k := f_k \partial_k$ for $k \in \{1, \dots, n+c\}$.

Its module of global sections can be described simply by forgetting ∂_{f_k} , i.e. we have the following description

$$\frac{\mathbb{C}[z, f_1^\pm, \dots, f_{n+c}^\pm, q_1^\pm, \dots, q_r^\pm] \langle z^2 \partial_z, z \partial_{q_1}, \dots, z \partial_{q_r} \rangle}{((Q_l)_{l \in \mathbb{L}} + (\widehat{E}))}. \quad (67)$$

Notice that the map \bar{q} can be factorized as $\theta \circ i_\theta$ with

$$\begin{aligned} i_\theta : \mathcal{KM} &\longrightarrow F \times \mathcal{KM}, \\ (q_1, \dots, q_r) &\mapsto (1, \dots, 1, q_1, \dots, q_r) \end{aligned}$$

Thus the inverse image of (67) with respect to i_θ is given by

$$\frac{\mathbb{C}[z, q_1^\pm, \dots, q_r^\pm] \langle z^2 \partial_z, z \partial_{q_1}, \dots, z \partial_{q_r} \rangle}{((Q_l)_{l \in \mathbb{L}} + (\widehat{E}))},$$

which is exactly the definition of the module \mathcal{P} from Theorem 6.7.

Concerning the second isomorphism, the associated sub- $R_{\mathbb{C}_z \times F \times \mathcal{KM}}$ -module corresponding to $\widehat{\mathcal{K}}_{\mathcal{N}}$ from Lemma 6.2 can be described by

$$\{P \in \mathbb{C}[z, f_1^\pm, \dots, f_{n+c}^\pm, q_1^\pm, \dots, q_r^\pm] \langle z^2 \partial_z, z \partial_{q_1}, \dots, z \partial_{q_r} \rangle \mid \exists p \in \mathbb{Z}, k \in \mathbb{N} \text{ s.t. } \prod_{i=0}^k \widehat{C}_{top}^{p+i} P \in ((Q_l)_{l \in \mathbb{L}} + (\widehat{E}))\},$$

where

$$\begin{aligned} \widehat{C}_{top}^k &:= \prod_{i=m+1}^{m+c} \left(\left(\sum_{j=1}^{n+c} c_{ij} f_j \partial_j + \sum_{a=1}^r l_{ia} q_a \partial_a \right) + l \right), \\ &= \prod_{i=m+1}^{m+c} \left(\left(\sum_{j=1}^{n+c} c_{ij} f_j \partial_j + \widehat{D}_i \right) + k \right) \end{aligned}$$

for $k \in \mathbb{Z}$. It is easy to see that its inverse image under $(id_{\mathbb{C}_z \times i_\theta})$ is given by

$$\{P \in \mathbb{C}[z, q_1^\pm, \dots, q_r^\pm] \langle z^2 \partial_z, z \partial_{q_1}, \dots, z \partial_{q_r} \rangle \mid \exists p \in \mathbb{Z}, k \in \mathbb{N} \text{ s.t. } \prod_{i=0}^k \widehat{C}_{top}^{p+i} P \in ((Q_l)_{l \in \mathbb{L}} + (\widehat{E}))\},$$

which is exactly the definition of the ideal K in Definition 6.8. Thus, the second isomorphism follows. \square

Combining Proposition 6.10, Theorem 6.7 and Propositions 6.4 and 6.5, we obtain the following mirror statement.

Theorem 6.11. *Let X_Σ and $\mathcal{L}_1, \dots, \mathcal{L}_c$ as in Theorem 6.7. Consider the affine resp. non-affine Landau-Ginzburg models $\pi_1 = (\bar{F}, \underline{q}) : X^{aff} \times \mathcal{KM}^\circ \rightarrow \mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ$, $\pi_2 : S \times \mathcal{KM}^\circ \rightarrow \mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ$ and $\Pi : \mathcal{Z}_X^\circ \hookrightarrow \mathbb{Z}^\circ \xrightarrow{\alpha} \mathbb{C}_{\lambda_0} \times \mathcal{KM}^\circ$ associated to $(X_\Sigma, \mathcal{L}_1, \dots, \mathcal{L}_c)$. Let $B_\varepsilon^* \subset \mathcal{KM}^\circ$ be the punctured ball from Theorem 6.7. Then there are isomorphisms of $\mathcal{D}_{\mathbb{C}_z \times B_\varepsilon^*}$ -modules*

$$FL_{\mathcal{KM}^\circ}^{loc} (\mathcal{H}^0 \pi_{2 \dagger} \mathcal{O}_{S \times \mathcal{KM}^\circ})|_{\mathbb{C}_z \times B_\varepsilon^*} \cong (id_{\mathbb{C}_z} \times \text{Mir})^* (\text{QDM}(X_\Sigma, \mathcal{E}))|_{\mathbb{C}_z \times B_\varepsilon^*} \otimes \mathcal{O}_{\mathbb{C}_z \times B_\varepsilon^*} (*(\{0\} \times B_\varepsilon^*)),$$

$$FL_{\mathcal{KM}^\circ}^{loc} (\mathcal{H}^0 \alpha_+ \mathcal{M}^{IC}(\mathcal{Z}_X^\circ))|_{\mathbb{C}_z \times B_\varepsilon^*} \cong (id_{\mathbb{C}_z} \times \text{Mir})^* (\overline{\text{QDM}}(X_\Sigma, \mathcal{E}))|_{\mathbb{C}_z \times B_\varepsilon^*} \otimes \mathcal{O}_{\mathbb{C}_z \times B_\varepsilon^*} (*(\{0\} \times B_\varepsilon^*))$$

and an isomorphism of $\mathcal{R}_{\mathbb{C}_z \times B_\varepsilon^*}$ -modules

$$\left(H^{n+c}(\Omega_{X^{\text{aff}} \times \mathcal{K}\mathcal{M}^\circ / \mathcal{K}\mathcal{M}^\circ}(\log D)[z], zd - d\tilde{F}) \right)'_{|\mathbb{C}_z \times B_\varepsilon^*} \cong (\text{id}_{\mathbb{C}_z} \times \text{Mir})^* (\text{QDM}(X_\Sigma, \mathcal{E}))_{|\mathbb{C}_z \times B_\varepsilon^*}.$$

The following corollary is the promised Hodge theoretic application of the above main theorem.

Corollary 6.12. *There exists a variation of non-commutative pure polarized Hodge structures $(\mathcal{F}, \mathcal{L}_Q, \text{iso}, P)$ on $\mathcal{K}\mathcal{M}^\circ$ (see [KKP08], [HS10] or [Sab11] for the definition) such that*

$$\mathcal{F} \otimes \mathcal{O}_{\mathbb{C}_z \times B_\varepsilon^*} (*(\{0\} \times B_\varepsilon^*)) \cong (\text{id}_{\mathbb{C}_z} \times \text{Mir})^* (\overline{\text{QDM}}(X_\Sigma, \mathcal{E}))_{|\mathbb{C}_z \times B_\varepsilon^*} \otimes \mathcal{O}_{\mathbb{C}_z \times B_\varepsilon^*} (*(\{0\} \times B_\varepsilon^*)). \quad (68)$$

Proof. Using Theorem 6.11, this is a direct consequence of [Sai88, Théorème 1] and [Sab08, Corollary 3.15]. \square

It would of course be desirable to remove the use of the localization functor $- \otimes \mathcal{O}_{\mathbb{C}_z \times B_\varepsilon^*} (*(\{0\} \times B_\varepsilon^*))$ from the above theorem. We conjecture that the corresponding statement still holds, however, we cannot give a complete proof of this for the moment as we are not able to control the Hodge filtration on $\mathcal{M}^{IC}(\mathcal{Z}_X^\circ)$. More precisely, we expect the following to be true.

Conjecture 6.13. 1. *Write $F_\bullet^H \mathcal{H}^0 \alpha_+ \mathcal{M}^{IC}(\mathcal{Z}_X^\circ)$ for the Hodge filtration on the pure Hodge module (see [Sai88, Théorème 1]) $\mathcal{H}^0 \alpha_+ \mathcal{M}^{IC}(\mathcal{Z}_X^\circ)$, which has weight $m + n + 2c$. Let $F_\bullet^H[\partial_{\lambda_0}^{-1}]$ be the saturation of F_\bullet^H as in the proof of Lemma 6.4 and write G_\bullet^H for the induced filtration on $\text{FL}_{\mathcal{K}\mathcal{M}^\circ}(\mathcal{H}^0 \alpha_+ \mathcal{M}^{IC}(\mathcal{Z}_X^\circ))$. Then under the isomorphism of Proposition 6.5, 2., we have that*

$$G_{\bullet-(m+n+2c)}^H \text{FL}_{\mathcal{K}\mathcal{M}^\circ}(\mathcal{H}^0 \alpha_+ \mathcal{M}^{IC}(\mathcal{Z}_X^\circ)) \cong z^\bullet \cdot {}_0\mathcal{Q}\mathcal{M}_{A'}^{IC}.$$

Notice that the bundle \mathcal{F} which was used in the isomorphism from corollary 6.12 is nothing but the object $G_{-(m+n+2c)}^H \text{FL}_{\mathcal{K}\mathcal{M}^\circ}(\mathcal{H}^0 \alpha_+ \mathcal{M}^{IC}(\mathcal{Z}_X^\circ))$.

2. *The isomorphism (68) holds without localization, i.e., there is an isomorphism of $\mathcal{R}_{\mathbb{C}_z \times B_\varepsilon^*}$ -modules*

$$\left(G_{-(m+n+2c)}^H \text{FL}_{\mathcal{K}\mathcal{M}^\circ}(\mathcal{H}^0 \alpha_+ \mathcal{M}^{IC}(\mathcal{Z}_X^\circ)) \right)'_{|\mathbb{C}_z \times B_\varepsilon^*} \cong (\text{id}_{\mathbb{C}_z} \times \text{Mir})^* (\overline{\text{QDM}}(X_\Sigma, \mathcal{E}))_{|\mathbb{C}_z \times B_\varepsilon^*}.$$

As a consequence, the reduced quantum \mathcal{D} -module underlies a variation of non-commutative Hodge structures.

Comparing Theorem 6.11 with Lemma 6.4 one may wonder whether the module $\text{FL}_{\mathcal{K}\mathcal{M}^\circ}^{\text{loc}}(\mathcal{H}^0 \pi_2 + \mathcal{O}_{S \times \mathcal{K}\mathcal{M}^\circ})$ also has an interpretation as a mirror object. This is actually the case, namely, it corresponds to the so-called *Euler*⁻¹-twisted quantum \mathcal{D} -module (whereas the object $\text{QDM}(X_\Sigma, \mathcal{E})$ from Definition 4.3 would be the *Euler*-twisted quantum \mathcal{D} -module in this terminology). The *Euler*⁻¹-twisted quantum \mathcal{D} -module encodes the so-called **local Gromov-Witten invariants** of the dual bundle \mathcal{E}^\vee and is denoted by $\text{QDM}(\mathcal{E}^\vee)$ (see [Giv98, Theorem 4.2]). The forthcoming paper [IMM12] will show the existence of a non-degenerate pairing between $\text{QDM}(X_\Sigma, \mathcal{E})$ and $(\text{id}_{\mathbb{C}_z} \times f)^* \text{QDM}(\mathcal{E}^\vee)$ (non-equivariant limit of the quantum Serre theorem) where $f \in \mathbb{C}[[H^*(X_\Sigma, \mathbb{C})^\vee]]^n$ is a certain coordinate change. However, in this theorem, all objects are defined on the total cohomology space, i.e., correspond to the big (twisted) quantum product. A rather easy calculation shows that f restricts to a (formal) map $H^2(X_\Sigma, \mathbb{C}) \rightarrow H^0(X_\Sigma, \mathbb{C}) \oplus H^2(X_\Sigma, \mathbb{C})$. Using these constructions, we conjecture the following additional mirror statement. Its proof is done by comparing the duality statements from section 5 with the non-equivariant limit of quantum Serre theorem from [IMM12]. We postpone the details to a subsequent paper.

Conjecture 6.14. *Consider the situation of Theorem 6.11, in particular, let $\mathcal{E} := \bigoplus_{j=1}^c \mathcal{L}_j$. As \mathcal{L}_j are nef bundles and hence globally generated, also \mathcal{E} is globally generated and therefore convex. Let $\text{QDM}(\mathcal{E}^\vee)$ be the (*Euler*⁻¹)-twisted quantum \mathcal{D} -module governing local Gromov-Witten invariants, that is, integrals over the moduli space $\overline{\mathcal{M}}_{0,l,d}(\mathcal{V}(\mathcal{E}^\vee))$ of stable maps to the total space $\mathcal{V}(\mathcal{E}^\vee)$ (notice that $\overline{\mathcal{M}}_{0,l,d}(\mathcal{V}(\mathcal{E}^\vee))$ is compact unless $d = 0$). Then there is some convergency neighborhood $B_{\varepsilon'}^*$, an isomorphism of $\mathcal{D}_{\mathbb{C}_z \times B_{\varepsilon'}^*}$ -modules*

$$\text{FL}_{\mathcal{K}\mathcal{M}^\circ}^{\text{loc}}(\mathcal{H}^0 \pi_2 + \mathcal{O}_{S \times \mathcal{K}\mathcal{M}^\circ})_{|\mathbb{C}_z \times B_{\varepsilon'}^*} \cong (\text{id}_{\mathbb{C}_z} \times \text{Mir}')^* \left(\text{QDM}(\mathcal{E}^\vee)_{|\mathbb{C}_z \times B_{\varepsilon'}^*} \otimes \mathcal{O}_{\mathbb{C}_z \times B_{\varepsilon'}^*} (*(\{0\} \times B_{\varepsilon'}^*)) \right)$$

and an isomorphism of $\mathcal{R}_{\mathbb{C}_z \times B_\varepsilon^*}$ -modules

$$H^{n+c}(\Omega_{X_{\text{aff}}^\bullet \times \mathcal{KM}^\circ / \mathcal{KM}^\circ}(\log D)[z], zd - d\tilde{F})|_{\mathbb{C}_z \times B_\varepsilon^*} \cong (\text{id}_{\mathbb{C}_z} \times \text{Mir}')^* \left(\text{QDM}(\mathcal{E}^\vee)|_{\mathbb{C}_z \times B_\varepsilon^*} \right).$$

Here Mir' is some base change involving both f and Mir . In view of [Giv98, corollary 4.3], one may conjecture further that Mir' is the identity if the number c of line bundles defining the bundle \mathcal{E} is strictly bigger than 1.

Remark: The following consideration shows that the main Theorem 6.11 can also be considered as a generalization of mirror symmetry for Fano manifolds themselves, as presented in our previous paper (see [RS10, Proposition 4.10]). Namely, let us consider the case where the number c of line bundles on the toric variety X_Σ is zero. Then we have $A' = A$, and the duality morphism ϕ from Theorem 5.4 is

$$\phi : \mathcal{M}_{A''}^{-(c+1,0,1)} = \mathcal{M}_{A''}^{(-1,0)} \longrightarrow \mathcal{M}_{A''}^{(0,0,0)} = \mathcal{M}_{A''}^{(0,0)}$$

and is induced by right multiplication by ∂_{λ_0} . In particular, the induced morphism $\widehat{\phi}$ is simply the identity on $\widehat{\mathcal{M}}_{A'}^{(0,0)}$. In particular, we have that $\text{im}(\widehat{\phi}) \cong \widehat{\mathcal{M}}_{A'}^{(0,0)}$ so that $\mathcal{QM}_{A'}^{IC} \cong \mathcal{QM}_{A'}$ and ${}^0\mathcal{QM}_{A'}^{IC} \cong {}^0\mathcal{QM}_{A'}$. On the other hand, the reduced quantum \mathcal{D} -module $\text{QDM}(X_\Sigma, \mathcal{E})$ is nothing but the quantum \mathcal{D} -module of the variety X_Σ , so that we deduce from Theorem 6.11 that we have an isomorphism of $\mathcal{D}_{\mathbb{C}_z \times B_\varepsilon^*}$ -modules

$$\text{FL}_{\mathcal{KM}^\circ}^{\text{loc}} \left(\mathcal{H}^0 \pi_2 + \mathcal{O}_{S \times \mathcal{KM}^\circ} \right)|_{\mathbb{C}_z \times B_\varepsilon^*} \cong (\text{id}_{\mathbb{C}_z} \times \text{Mir}')^* \left(\text{QDM}(X_\Sigma)|_{\mathbb{C}_z \times B_\varepsilon^*} \otimes \mathcal{O}_{\mathbb{C}_z \times B_\varepsilon^*} (*(\{0\} \times B_\varepsilon^*)) \right).$$

One easily sees that we have an even more precise statement, namely, the third assertion of Theorem 6.11 simplifies in this case to an isomorphism of $\mathcal{R}_{\mathbb{C}_z \times B_\varepsilon^*}$ -modules

$$H^n(\Omega_{S \times \mathcal{KM}^\circ / \mathcal{KM}^\circ}^\bullet[z], zd - d\tilde{F})|_{\mathbb{C}_z \times B_\varepsilon^*} \cong (\text{id}_{\mathbb{C}_z} \times \text{Mir}')^* \left(\text{QDM}(X_\Sigma, \mathcal{E})|_{\mathbb{C}_z \times B_\varepsilon^*} \right).$$

This isomorphism is the restriction of the isomorphism in [RS10, Proposition 4.10] to $\mathbb{C}_z \times B_\varepsilon$ (see also [Iri09, Proposition 4.8]), notice that the neighborhood B_ε is called W_0 in [RS10]. Hence we see that our main Theorem 6.11 contains in particular the mirror correspondence for smooth toric nef manifolds, at least on the level of $\mathcal{R}_{\mathbb{C}_z \times B_\varepsilon}$ -modules.

One may conclude from the above observation that Landau-Ginzburg models, either affine or compactified, appear to be the right point of view to study various type of mirror models of (the quantum cohomology of) smooth projective manifolds, including Calabi-Yau, Fano and more generally nef ones. The recent preprint [GKR12] where varieties of general types and their mirrors are investigated, also seem to confirm this observation. It would certainly be fruitful to apply our methods to varieties with positive Kodaira dimension to refine the results from loc.cit.

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