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October 14, 2023

Mathematics and Information, Exercise sheet 5

Problem 1: (*4 points*)

Let X, Y be two real valued random variables with correlation ρ.

a) Determine the eigenvalues and eigenvectors of their correlation matrix!

b) Compare the eigenvectors to the vectors b_i, whose j^{th} component is $\cos\left(\frac{(2j-1)(i-1)\pi}{4}\right)$!

c) Find two unit vectors $\binom{a_1}{a_2}$ and $\binom{b_1}{b_2}$ such that $a_1X + a_2Y$ and $b_1X + b_2Y$ are uncorrelated!

Problem 2: (*7 points*)

A bet is offered on the outcome of an experiment leading to one of two possible results A and B. Whoever guesses the right result gets his stakes doubled, all others loose their stake. You happen to know that as a matter of fact outcome A has a probability p with $\frac{1}{2} < p < 1$. Your available capital for betting is C_0.

a) What is your expected return if you set all your capital on A ?

b) Now assume that the experiment is repeated n times, and the outcomes of the different repetitions are independent of each other. You decide to set a fixed proportion of your available capital on A each time. What is your expected return after n repetitions?

c) Which proportion of the available capital should you choose to maximize this expectation?

d) What is the probability that your final capital is at least equal to that expected return?

e) What is the probability that you loose all your capital?

f) Alternatively you could choose the proportion in such a way that you maximize the expected growth rate. What is your expected capital after n repetitions then?

g) What is the probability of a total loss in this case!

Problem 3: (*4 points*)

For the game *Lotto light,* invented for this exercise, every player chooses a number from one to eight. Every week such a number is drawn by a fair process, where each number has equal likelihood to be drawn. All the stakes are accumulated and equally distributed among the winners. Empirical investigations showed that betters have different probabilities to choose one of the numbers; the probability to choose i is p_i given in the following table:

i	1	2	3	4	5	6	7	8
p_i	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$

a) What is the expected return for each of the eight possible bets?

b) What is the expected return of a random bet?

c) Let's assume somebody plays *Lotto light* every week, starting with a fixed capital set aside for betting. Which number should he choose and which proportion of the available capital should he bet in order to achieve an optimal growth rate?

Problem 4: *(5 points)*

Suppose on a race course you can bet on the winner of every race. If you pick the winning horse, your stake is multiplied by some odds and returned to you, otherwise you loose your stake. Assume that different bookmakers offer different odds for the same horse. For the i^{th} horse the maximal odds are o_i, and $S = \sum 1/o_i < 1$. In this case a so called *Dutch book* strategy is possible: For each horse i you set a fraction $b_i = 1/So_i$ of your capital on it, placing your bet with the bookmaker offering the best odds for it.

a) What is your gain or loss with this portfolio, if the i^{th} horse wins?

b) Suppose, there are only two horses. One bookmaker offers $o_1 = 1.5$ on the first horse, another offers $o_2 = 6$ on the second. Determine the *Dutch book* portfolio!

c) Assume that both horses are equaliy likely to win. Which doubling rate does one get with the *Dutch book?*

Can be submittel till Wednesday, October 18, 2023 at 15.30h