Wolfgang K. Seiler B6, B4.08
Tel. 2515

September 14, 2023

Mathematics and Information, Exercise sheet 1

Problem 1: (*5 points*)

a) A message source produces signs form an alphabet A consisting of six letters with respective probabilities $1/2, 1/4, 1/8, 1/16$, and twice $1/32$. Compute its Shannon entropy!

b) Construct a binary encoding of these letters such that any sequence of letters from A has a unique encoding, and the expected value of the code length is as small as possible!

c) What changes in *a)* and *b)*, if all elements of A have equal probability?

Problem 2: (*2 points*)

a) Show that the intersection of two convex sets $A, B \subseteq \mathbb{R}^n$ is again convex!

b) What about their union $U \cup B$ und there symmetric difference $A \Delta B$?

Problem 3: (*4 points*)

a) An alphabet A contains n letters; the probability of each letter is at least $1/2n$. Compute the minimum and the maximum for the entropy of a source using such an alphabet!

b) What happens, if instead we know that each letter occurs with a probability $p \leq 2/n$?

Problem 4: (*5 points*)

Let X, Y be random variables with values in $A = \{0, 1\}$ and joint probability distribution $p(0,0) = \frac{1}{2}$, $p(0,1) = p(1,1) = \frac{1}{4}$ and $p(1,0) = 0$.

a) Find the probability distributions p_X, p_Y of X and Y!

b) Determine $H(X)$, $H(Y)$, $H(X,Y)$, $H(X|Y)$, $H(Y|X)$ and $I(X;Y)$!

c) Compute the Kullback-Leibler distances $d(p_X \| p_Y)$ and $d(p_Y \| p_X)$!

Problem 5: (*4 points*)

a) A fair dice is thrown; the random variable X, Y, Z with values in $\{1, 2, 3, 4, 5, 6\}$ give the numbers on top, at the bottom and on the front side. Compute the mutual informations $I(X;Y)$, $I(X;Z)$ and $I(Z;Y)$!

Hint: The numbers on opposite sides of a dice always add up to seven.

b) Determine the conditional mutual informations $I(X;Y|Z)$ and $I(X;Z|Y)$!

Can be submittel till Wednesday, September 20, 2023 at 15.30h