Sheaf Cohomology

Notes by Sven Balnojan to script by Andreas Gathmann

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0.1 Introduction

I quickly walk through the very basics of sheaf cohomology for sheaves that generalize vector bundles. I mention how this is related to derived functors and mostly work with facts & examples. I mention Serre duality mostly because I need it in a seminar talk, but it's an extremely important feature.

1 Generalizing Vector Bundles

I'I'm assuming you've heard all the terms that I now mention. So how do quasicoherent sheaves generalize vector bundles again? $\mathcal{O}_X$-modules form a nice (i.e. abelian) category, so do e.g. vector spaces. But locally free sheaves (= vector bundles) do not (see [Vakil2013, p. 362] for all this + counter example); We can enlarge that category to an abelian one however. So we have

$$\begin{matrix} \mathcal{O}_X - \text{modules} \\ \text{(abelian)} \end{matrix} \supset \begin{matrix} \text{Qcoh}(X) \\ \text{(abelian)} \end{matrix} \supset \begin{matrix} \text{coh}(X) \\ \text{(abelian)} \end{matrix} \supset \begin{matrix} \text{vector bundles} \\ \text{(not abelian)} \end{matrix}.$$

So we enlarge the category and get an abelian one. Is this the smallest abelian enlargement which is "local"? Turns out no, the category of coherent sheaves is even smaller.

Remark 1. (i) The category is *abelian* so we can form the derived category, meaning we can consider the objects as "complexes of objects".

(ii) The category of vector spaces is abelian, but it's also *triangulated* $\implies$ *semisimple.* Semisimple categories are kind of boring (that every exact sequence of vector spaces splits is the incarnation of the semisimplicity), from the homological point of view, any complex is quasi-isomorphic to a complex with $H^i = 0$ for all $i > 0$.

(iii) The next simple categories have complexes (in their derived categories) with $H^i = 0$ for all $i > 1$. Those are called *hereditary* categories.

1.1 (Quasi-) Coherent Sheaves

The class of $\mathcal{O}_X$-modules is usually too large to study. Any ring R determines an affine scheme spec$R = X$ with structure sheaf $\mathcal{O}_X$. Any R-module M determines a sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules on $X = \text{spec}R$. Is every sheaf of $\mathcal{O}_X$-modules of this form?

No, the ones that are are called *quasicoherent.* The ones that are associated to a finitely generated R-module are called *coherent.*

1.2 Examples

We have a good idea on how coherent sheaves on the projective line look like.

Theorem 2. *Any coherent sheaf on $\mathbb{P}^1(\mathbf{k})$ is the finite direct sum of line bundles and degree one skyscraper sheaves (as not every coherent sheaf is locally free we need the skyscraper sheaves; that's the point we want to generalize vector bundles).*

Let's look at an example of a non-coherent sheaf.

Example 3. We want to construct a non-coherent sheaf. So we fix a affine scheme $X = \mathbb{A}^1_k$; We are look for a sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules (= an assignment: set $\mapsto \mathcal{O}_X$-module) that is not induced by some $\mathcal{O}_X$-module M.

If this were the case, then we'd know that $\tilde{M}(X) = M$; If we construct a non-trivial sheaf with no non-trivial global sections this then equals $M = 0$; Which is sufficient because the sheaf induced by the module $M = 0$ is the trivial sheaf. The sheaf in question is simply (the sheaf associated to the presheaf defined by):

$$U \mapsto \begin{cases} 0 & 0 \in U \\ \mathcal{O}_X(U) & 0 \notin U \end{cases}.$$

2 Sheaf Cohomology

Why do we care about sheaf cohomology in this talk? For two reasons.

- Serre duality is an important feature of the categories we described above (coh$X$ on wpls, but not wps) and it's formulated in terms of sheaf cohomology.
- (Related) We have an equivalence to a category given by Ext^i which is just a derived functor. Indeed for the categories we are looking at, if we derive from the functor of global sections, we get the sheaf cohomology functor, so to determine Ext^i we can work with sheaf cohomology (as already mentioned in a talk before).

2.1 Serre Duality

Why is Serre duality important? Serre duality is a "pretty" property relating two cohomology classes (a special case being the Poincaré duality which lets us compute cohomology classes in terms of homology classes).

It states roughly: there is a unique ω_X (a line bundle = dualizing sheaf = canonical bundle; unique up to isomorphism) such that for all $\mathcal{F} \in \mathrm{coh}(X)$, $\forall i$, there is a natural isomorphism

$$(H^i(X, \mathcal{F}^\vee \otimes \omega_X) \cong) \quad \mathrm{Ext}^i(\mathcal{F}, \omega_X) \cong H^{n-i}(X, \mathcal{F})^*.$$

We start with a couple of reminders of basic algebraic geometry.

2.2 Twisting Sheaves

Take a projective variety $X \subset \mathbb{P}^n$, it's homogeneous coordinate ring $S(X)$ (with grading by degree, pieces denoted by $S^{(d)}$) and consider the localization on the non-zero homogenous elements. It has a grading, so we can simply define the nth graded piece

$$K(n) = \{\frac{f}{g}; f \in S^{(d+n)}, g \in S^{(d)}, \text{ some } d \geq 0 \text{ and } g \neq 0\}$$

and we define the twisting sheaf $\mathcal{O}_X(n)$ by

$$\mathcal{O}_X(n)(U) = \bigcap_{P \in U} \{\frac{f}{g} \in K(n); g(P) \neq 0\}.$$

Example 4. (a) $\mathcal{O}_{\mathbb{P}^n}(0) = \mathcal{O}_{\mathbb{P}^n}$.

(b) $\mathcal{O}_{\mathbb{P}^n}(-1)$ is for $n = 1$ e.g. with coordinates x_0, x_1

$$\frac{1}{x_0} \in \mathcal{O}_{\mathbb{P}^1}(-1)(\{(x_0 : x_1); x_0 \neq 0\})$$

(c) locally $\mathcal{O}_X(n)$ is isomorphic to $\mathcal{O}_X$. In (b) e.g. we have the maps (on U, upper indices mean degrees):

$$\frac{f^1}{g^0} \mapsto \frac{1}{x_0} \cdot \frac{f^1}{g^0} \text{ and } \frac{f^1}{g^1} \mapsto \frac{f^1}{g^1} \cdot \frac{1}{x_0}$$

(d) however the sheaves are not globally isomorphic. If they where, then the global sections would be the same. But $\Gamma(\mathbb{P}^n_k, \mathcal{O}_X) \cong k$ while $\Gamma(\mathbb{P}^n_k, \mathcal{O}_X(n))$ is either empty for $n < 0$ or defined by homogeneous polynomials of degree n (that's pretty evident, the twisting is really a shifting of the grading; "Shifting k by -1 produces non-holomorphic functions, while shifting k by $+1$ produces polynomials of degree 1").

Some more interesting properties of twisting sheaves are the following.

Example 5. (i) Let $X = \mathbb{P}^n$. We have $\mathcal{O}_X(d) \otimes \mathcal{O}_X(l) = \mathcal{O}_X(d+l)$ using the isomorphism

$$\frac{f}{g} \otimes_{\mathcal{O}_X(U)} \frac{\tilde{f}}{\tilde{g}} \mapsto \frac{f\tilde{f}}{g\tilde{g}}.$$

(this defines a presheaf; associate to it the sheaf). This is enough a the tensor product of the modules is generated by the images of the tensor product $\otimes_{O_X(U)}$.

(ii) We have $\mathcal{O}_X(d)^\vee = \mathcal{O}_X(-d)$. The dual sheaf is defined (again by a presheaf) as $U \mapsto Hom_{\mathcal{O}_X(U)}(\mathcal{O}_X(d), \mathcal{O}_X)$. But the $\mathcal{O}_X$-linear homomorphisms are given exctly by multiplication with sections of $\mathcal{O}_X(-d)$ (multiplication obviously gives an element in $\mathcal{O}_X$).

2.3 Differential Forms

If $Y = \mathrm{Spec}\mathbf{k}$ is a point, then we write Ω_X for the *sheaf of differential forms*. In that case, for a smooth manifold this coincides with usual differential forms (we could also define the sheaf of relative differential forms and make this a special case). If X is a smooth scheme, then (if and only if) is Ω_X locally free of dimension n. So it's a vector bundle, the *cotangent bundle*. We define

Definition 6. The *canonical bundle* of a smooth n-dimensional scheme X is the bundle $\omega_X := \wedge^n \Omega_X$ (top exterior power of 1-forms).

(It's canonical because it's canonically defined for any scheme and thus gives a method to compare schemes up to isomorphism).

Example 7. ω_X on $\mathbb{P}^n$. The canonical bundle = cotangent bundle of $\mathbb{P}^n$ is determined by an exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to^{\varphi_1} \mathcal{O}(-1)^{\oplus(n+1)} \to^{\varphi_2} \mathcal{O} \to 0$$

then a "basic AG theorem" yields $\omega_X \cong \wedge^{n+1}(\mathcal{O}(-1)^{\oplus(n+1)}) \cong \mathcal{O}(-n-1)$.

Example 8. (i) (Of Serre duality, even though we haven't defined the sheaf cohomology yet) Determine $H^{n-1}(X, \mathcal{O}_X^\vee \otimes \omega_X)$. $\mathcal{O}_X^\vee$ is locally free, and $\omega_X \cong \mathcal{O}(-n-1)$. So we have

$$H^{n-1}(X, \mathcal{O}_X^\vee \otimes \omega_X) = Ext^{n-1}(\mathcal{O}_X, \mathcal{O}(-n-1)).$$

By Serre duality the last expression is isomorphic to $H^1(X, \mathcal{O}_X)^*$. If X is affine, then this is (dual to) zero; If X is projective and 1 dimensional, then $H^1(X, \mathcal{O}_X) =$... . In any case, this is a lower cohomology group and simpler to calculate.

The other way around we could try to determine $H^{n-1}(X, \mathcal{O}_X)$; First determine $H^{n-1}(X, \mathcal{O}_X)$ by the stuff above, we only need to do $H^1(X, \mathcal{O}_X^\vee \otimes \mathcal{O}(-n-1))$ which again, is doable.

(ii) ??? (A doable calculation could be done for $\mathbb{P}^1$ or other curves)

2.4 Sheaf Cohomology

We explain the statement above in slightly more detail now. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of quasicoherent sheaves on some scheme X. If $X = SpecR$ is an affine scheme, then $\Gamma : qcoh(X) \to Mod(R)$ is an exact functor (and an equivalence of categories).

However if $X = ProjS$ is projective, then Γ may not be exact.

Example 9. Let $X \subset \mathbb{P}^n$ be some smooth hypersurface of degree d. Denote the inclusion by i. We use the exact sequence for the cotangent sheaf to determine the pullback bundles global sections (= trivial). We then use the relative differential form sequence (which includes the pullback global sections!) to show that this sequence then is not exact.

The cotangent sheaf is even defined by the exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O} \to 0 \text{ pulling back + global sections yields}$$
$$0 \to \Gamma(i^*\Omega_{\mathbb{P}^n}) \to \Gamma(\mathcal{O}(-1)^{\oplus(n+1)}) \to \Gamma(\mathcal{O}) \to 0$$

since $\mathcal{O}(-1)$ has no global sections (remember, global sections of $\mathcal{O}(d)$ are "defined by degree d polynomials"), the pullback bundle doesn't either (as it's an exact sequence!). On the other hand we have

$$0 \to \mathcal{O}_X(-d) \to i^*\Omega_{\mathbb{P}^n} \to \Omega_X \to 0 \text{ taking global sections}$$
$$0 \to \Gamma(\mathcal{O}_X(-d)) \to \Gamma(i^*\Omega_{\mathbb{P}^n}) \to \Gamma(\Omega_X) \to 0 \text{ where the last term is i.g. not trivial!}$$

So we conclude, the sequence is not exact and Γ is not exact.

Remark 10. A rough sketch of derived functors: We get a family of functors $R^i\Gamma$ by postulating that we want to extend sequences under this functor. So we want the functor to have the following exact sequence

$$0 \to \Gamma(\mathcal{F}_1) \to \Gamma(\mathcal{F}_2) \to \Gamma(\mathcal{F}_3) \to R^1\Gamma(\mathcal{F}_1) \to R^1\Gamma(\mathcal{F}_2) \to R^1\Gamma(\mathcal{F}_3) \to R^2\Gamma(\mathcal{F}_1) \to R^2\Gamma(\mathcal{F}_2)$$

in that way, the derived functor measures the exactness of our functor. If $R^1\Gamma = 0$, then Γ is already exact. But in our case it's not (for projective schemes that is).

So we could introduce $R^i\Gamma(\mathcal{F}_l)$ as $H^i(X, \mathcal{F}_l)$.

That the following definition actually satisfies this and that it's independent of the cover (which as we are using Cech-cohomology isn't trivial) is something we don't discuss.

Definition 11. Let X be an arbitrary scheme an $\mathcal{F}$ be a quasicoherent sheaf on X. We define sequence of Abelian groups and homomorphisms $C^0(\mathcal{F}) \to^{d^0} C^1(\mathcal{F}) \to ...$ with $d^{p+1} \circ d^p = 0$ (complexes with boundary operators) as usual, and then set $H^p(X, \mathcal{F}) = \ker d^p / \mathrm{im} d^{p-1}$ and equal to zero for $p < 0$. (For X a scheme over a field k those are k-vector spaces).

(i) We set $C^p(\mathcal{F}) = \Pi_{i_0 < ... < i_p} \mathcal{F}(U_{i_0} \cap ... \cap U_{i_p})$
(ii) $(d^p\alpha)_{i_0,...,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,..,i_{k-1},i_{k+1},...,i_{p+1}} |_{U_{i_0} \cap ...}$.

Let's now do a bunch of important examples.

Example 12. (i) Let X and $\mathcal{F}$ be arbitrary. $H^0(X, \mathcal{F}) = \ker(d^0)$. The map d^0 is then

$$\underline{\alpha} : \alpha_i \in \mathcal{F}(U_i); \underline{\alpha} \mapsto \alpha_i - \alpha_j |_{U_i \cap U_j} .$$

This is zero if and only if we can glue the α_is together to a global section. So $H^0(X, \mathcal{F}) = \ker(d^0) = \Gamma(\mathcal{F})$.

(ii) Let X be affine. Then we can choose the affine cover $\{X\}$ of only one element (any cover can be trivially extended by empty sets); Then $C^{p \geq 2} = 0$ and thus $H^{p \geq 2}(X, \mathcal{F}) = 0$. This generalizes easily to a cover of size m. If X has a size m affine cover, then $H^{p>m}(X, \mathcal{F}) = 0$. In particular, if X is projective and of dimension n, then $H^{p>n}(X, \mathcal{F}) = 0$.

(iii) Take $X = \mathbb{P}^1$ and $\mathcal{F} = \mathcal{O}_X(-2)$. The only interesting cohomology group is H^1 (H^0 is given by (i) and the rest are 0 by (ii)). By what we said above we know $H^1(X, \mathcal{F}) =$ degree d polynomials (thus 0 for $d < 0$, $\cong k$ for $d = 0$).

(iv) $X = \mathbb{P}^n$ and $\mathcal{F} = \mathcal{O}_X(d)$ (just state the result). Let $S = k[x_0, ..., x_n]$ denote the (graded) coordinate ring of X. Then the only non trivial cohomology groups are for $i = 0$ and $i = n$ with

$$\bigoplus_{d \in \mathbb{Z}} H^n(X, \mathcal{F}) \cong \bigoplus_{d \in \mathbb{Z}} S_{-n-1-d} := S'.$$

(v) For $X = \mathbb{P}^n$ and $\mathcal{F} = \omega_X = \mathcal{O}(-n-1)$ in particular we have $H^n(X, \mathcal{F}) \cong S$.

(vi) (*Doing calculations with this new better functor and with the long exact sequence*) $X = \mathbb{P}^n$ and $\mathcal{F} = \Omega_X$. We can fit the cotangent bundle into the exact sequence (of sheaves) from before (the Euler sequence). Then we get a long exact cohomology sequence (that's the point of deriving the functor) and get a pretty statement about the cohomology groups. Recall the sequence

$$0 \to \Omega_X \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_X \to 0$$

Then using what we've learned before ($H^i(\mathcal{O}(-1)) = 0$ for all i and $H^0 = \Gamma$ or 0 for $\mathcal{O}_X$) we have

$$H^i(X, \Omega_X) \cong \begin{cases} \Gamma(\Omega_X) & i = 0 \\ 0 & i > 0 \end{cases} \quad h^i(X, \Omega_X) \cong \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases}.$$

3 Resources

Most of this is from the script by Andreas Gathmann [Gathmann2013]. The intro to sheaves of $\mathcal{O}_X$-modules is by Ravi Vakil [Vakil2013] and the homological considerations are from the book by Gelfand and Manin [GM1996].

References

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