$D^b coh(\mathbb{P}^1)$ or the Structure of $coh(\mathbb{P}^1)$

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13 January, 2017, Mannheim

Abstract

Understanding (the category of) coherent sheaves on projective space.

Contents

We want to understand the category $coh(\mathbb{P}^1)$ (to further understand the underlying variety $\mathbb{P}^1$). It's an abelian category. Those categories are best understood via their derived categories *(especially if, and we do, we care about cohomology. To define cohomology we need cokernels, so we need an abelian category.).* Why the derived category? Because it's better to consider the object with all it's resolutions, functors like Ext (which are important for the structure!) are most easily defined as derived functors and quasi-isomorphisms turn into isomorphism there.

The important example from a previous seminar (by Falko Gauss). Set $X = \mathbb{P}^1$ then we have an equivalence

$$D^b coh(X) \leftrightarrow^{\cong} D^b rep(\text{Kronecker quiver}).$$

$$E \mapsto Ext^{\bullet}(\mathcal{O}_X \oplus \mathcal{O}_X(1), E)$$

How we exactly should understand this example (the functor) we will describe below.

Remark 1. *Relevance.*

(i) The right hand side is much easier to think of.

(ii) The sheaf $\mathcal{T} = \mathcal{O}_X \oplus \mathcal{O}_X(1)$ contains all the complexity of the category.

Important reminder: The example uses that $Ext^{\bullet}(\mathcal{T}, \mathcal{T})$ is concentrated at degree 0 (zero for everything else).

0.1 Generalizing the Example

We sketch a theorem which generalizes the example; then we explain the terminology in the theorem, explain the example some more and finally very roughly mention how we got the Kronecker quiver for the above equivalence.

Theorem 2. *Let X be smooth projective, $\mathcal{D} = D^b coh(X)$ and $\mathcal{T}$ a tilting sheaf (as the $\mathcal{T}$ above, def. below) and $A := End_{\mathcal{O}_X}(\mathcal{T})$, then there is a functor F which induces a triangulated equivalence. This means the functor F (and a functor G) derived (RF and LG) is an equivalence between the derived categories. The functor is*

$$F := Hom_{\mathcal{O}_X}(\mathcal{T}, -) : Coh(X) \to mod(A^{op})$$

(The functor G is $- \otimes_A \mathcal{T}$).

Let's explain this with the example from above a little further. Let $X \in coh(\mathbb{P}^1)$ then we can form $F(\mathcal{T}, X) = Hom_{\mathcal{O}_X}(\mathcal{T}, X)$. The right derived functor of this functor is isomorphic to $Ext^i(X, -)$. So for that reason in the above example the functor is only written for elements E of $coh(X)$ considered as 0-complexes in $D^b coh(X)$.

0.2 Tilting Sheaf

So what is a tilting sheaf $\mathcal{T}$? It's a sheaf that essentially captures all the complexity of the category $coh(X)$ for some smooth projective scheme X (for affine schemes this is easy, see the example below). Set $A = End_{\mathcal{O}_X}(\mathcal{T})$ as above. Then we define

Definition 3. Some $\mathcal{T} \in coh(X)$ is a tilting sheaf, if it satisfies the following three axioms

(T1) A has finite global dimension = any module over A has a finite projective resolution.

(T2) $Ext^i_{\mathcal{O}_X}(\mathcal{T}, \mathcal{T}) = 0 \; \forall i > 0$

(T3) $\mathcal{T}$ generates $\mathcal{D} = D^b coh(X)$ as a triangulated category (that's different from the usual "generating set of objects" property)

Example 4. Why is $\mathcal{T} = \mathcal{O}_X \oplus \mathcal{O}_X(1)$ for $X = \mathbb{P}^1$ a good example of a tilting sheaf?

(T1) "Homs for line bundles only go upwards".

(T2) this is just the property that $Ext^\bullet(\mathcal{T}, \mathcal{T})$ is concentrated at degree 0.

(T3) this is the important part (as with exceptional sequences); Roughly we know that any coherent sheaf can be resolved by sheaves that are sums of line bundles. So we only need to show that all line bundles are "generated" by $\mathcal{T}$ which we get by "taking cones and shifts".

More important examples are

Example 5. (i) If X is affine, then $coh(X) \cong fpmod(A)$ (by taking global sections $\Gamma(X, -)$). For affine X this functor is exact, ($\implies$ (T2)), the tilting sheaf if $\mathcal{O}_X$.

(ii) For $X = \mathbb{P}^n$ the general tilting sheaf if $\mathcal{T} = \mathcal{O} \oplus \mathcal{O}(1) \oplus ... \oplus \mathcal{O}(n)$.

Even rougher, and without definitions, we now sketch important consequences. With that we can finally finish the example from above.

Corollary 6. *(i) If X, smooth projective, admits a tilting sheaf, then its Grothendieck (which can be endowed with a pairing called the Mukai pairing) is finitely generated and free (this leads to the notion of a generalized root system due to Takahashi et al.).*

(ii) If X, smooth projective, admits a tilting sheaf, and that tilting sheaf is a sum of line bundles, then those line bundles form an exceptional collection.

So in the example above (and in the generalized version) we get an exceptional sequence $(\mathcal{O}, \mathcal{O}(1))$ which lets us describe a quiver by setting the vertices as the elements of the sequence, the edges as the bases of the hom spaces (and we get relations from equality of morphisms).

So in the example above, we have two edges and we know that $Hom_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X(1)) \cong \mathcal{O}_X(1)$ and thus we get a Kronecker quiver (details in what F. Gauss did in the previous talk).

0.3 Serre Duality

The tilting sheaf in $coh(\mathbb{P}^n)$ is important for the structure. The second (and last) important property is the Serre duality (which holds).

It states roughly: there is a unique ω_X (a line bundle = dualizing sheaf = canonical bundle; unique up to isomorphism) such that for all $\mathcal{F} \in \mathrm{coh}(X)$, $\forall i$, there is a natural isomorphism

$$\mathrm{Ext}^i(\mathcal{F}, \omega) \cong H^{n-i}(X, \mathcal{F})^*.$$

Serre duality is a "pretty" property relating two cohomology classes (a special case being the Poincaré duality which lets us compute cohomology classes in terms of homology classes). Since we are in the category $coh(X)$ we have $Ext^i(\mathcal{F}, \mathcal{G}) \cong H^i(X, Hom(\mathcal{F}, \mathcal{G}))$. So Serre duality reduces the work to compute cohomology groups.

Example 7. For $X = \mathbb{P}^n$ we know the canonical bundle $\omega_X = \mathcal{O}(-n-1)$. By the theorem above we know the equivalence $D^b coh(X) \cong D^b rep(Q^{op})$ for some quiver Q (determined by the exceptional sequence induced by the tilting sheaf $\mathcal{T}$).

On the other hand we also know $D^b rep(Q^{op}) \cong mod - kQ$ where kQ is the path algebra over Q. From the latter category we know, since kQ is a principle ideal ring, that all objects have a projective resolution of length 2. So the homological dimension of that category is 1, so all those categories are hereditary (meaning $Ext^2(-,-) = 0$ for $i > 1$).

This is also in line with what we know about the sheaf cohomology of X with coefficients in the Serre twisting sheaves.

0.4 Perspective

Connecting to the last example we point out why weighted projective lines have gotten some attention recently. See for [CK2009] for the following comment: According to Happel here are only two classes of (connected) hereditary abelian categories that admit a tilting object (in the sense of a tilting sheaf as above); module categories over path algebras of quivers, and categories of coherent sheaves on weighted projective lines (projective spaces are trivial weighted projective lines). For any such category we have the aforementioned equivalence.

0.5 Resources

Most of the material is from a seminar by Hiro Lee Tanaka, see [Web01]. Some stuff about derived categories of quiver is from [TS2015], the last subsection refers to [CK2009]. Everything about homological algebra can be found in [GM1996].

The explicit structure of weighted projective lines, their structure sheaf and the explicit Serre duality can be found in the original work of [GL1987].

References

[GL1987] Geigle, W., & Lenzing, H. (1987). A class of weighted projective curves arising in representation theory of finite-dimensional algebras. In Singularities, representation of algebras, and vector bundles (Lambrecht, 1985) (Vol. 1273, pp. 265–297). Berlin, Heidelberg: Springer, Berlin. http://doi.org/10.1007/BFb0078849

[STK2014] Shiraishi, Y., Takahashi, A., & Wada, K. (2014, January 19). On Weyl Groups and Artin Groups Associated to Orbifold Projective Lines. arXiv.org.

[BR1986] Beltrametti, M., & Robbiano, L. (1986). Introduction to the theory of weighted projective spaces. Expositiones Mathematicae. International Journal for Pure and Applied Mathematics, 4(2), 111–162.

[CK2009] Chen, X.-W., & Krause, H. (2009, November 23). Introduction to coherent sheaves on weighted projective lines.

[TS2015] Tim Seynnaeve, 2015, http://www.math.uni-bonn.de/people/oschnuer/wise15/seminar/seynnaeve-DerivedCategoryOfADynkinQuiver.pdf

[Web01] http://math.harvard.edu/~hirolee/pdfs/280x-13-14-dbcoh.pdf

[GM1996] Gelfand, Manin - Methods of homological algebra. (1996). Gelfand, Manin - Methods of homological algebra.