Mappings from space to the plane

1. Linear maps (axonometries)

   a) orthogonal

   b) isometric

   c) dimetric

2. Perspectivic projections

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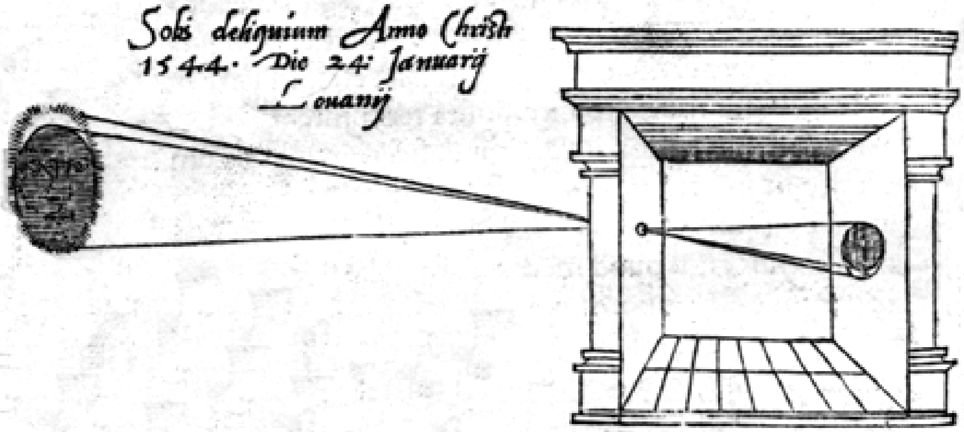

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Homogeneous coordinates

Affine mapping: $\vec{v} \mapsto A\vec{v} + \vec{b}$, where $A \in \mathbb{R}^{3\times 3}$ and $\vec{b} \in \mathbb{R}^3$.

Let $A^* = \begin{pmatrix} A & \vec{b} \\ \mathbf{0} & 1 \end{pmatrix}$, then

$$A^* \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \\ 1 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} u \\ v \\ w \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \vec{b}.$$

$$(x : y : z : u) = \left(\frac{x}{u}, \frac{y}{u}, \frac{z}{u}\right) \quad \textbf{if} \quad u \neq 0$$

otherwise: $(x, y, z, 0)$ =point at infinity

in direction of the line through $(0,0,0)$ and (x,y,z).

z-buffer makes screen three dimensional

matrix stacks contain several transformation matrices being multiplyed to give the final transformation

Geometrical objects

Polyhedrons

are given by their bounding polygones.

`glBegin( GL_POLYGON );`

`glColor3f(` r_1, g_1, b_1 `);`

`glNormal3f(` n_{x1}, n_{y1}, n_{z1} `);`

`glVertex3f(` x_1, y_1, z_1 `);`

`glColor3f(` r_2, g_2, b_2 `);`

`glNormal3f(` n_{x2}, n_{y2}, n_{z2} `);`

`glVertex3f(` x_2, y_2, z_2 `);`

$\ldots$

`glColor3f(` r_N, g_N, b_N `);`

`glNormal3f(` n_{xN}, n_{yN}, n_{zN} `);`

`glVertex3f(` x_N, y_N, z_N `);`

`glEnd();`

Subdivision for example by DELAUNAY triangulation

Parametric surfaces

Use **surfaces patches**

$$f\colon B \to \mathbb{R}^3; \quad (u,v) \mapsto f(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

$B \subseteq \mathbb{R}^2$ ist typycally a triangle or rectangle, f smooth.

curve patches

$$\gamma\colon [a,b] \to \mathbb{R}^3, \quad \gamma \text{ smooth}$$

Tangent vector $\gamma'(t) = \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \gamma_3'(t) \end{pmatrix}$

parameter lines through $f(u_0, v_0)$

$$\gamma_1\colon \begin{cases} (u_0 - \varepsilon, u_0 + \varepsilon) \to \mathbb{R}^3 \\ \qquad t \mapsto f(t, v_0) \end{cases}$$

and

$$\gamma_2\colon \begin{cases} (v_0 - \varepsilon, v_0 + \varepsilon) \to \mathbb{R}^3 \\ \qquad t \mapsto f(u_0, t) \end{cases}$$

The tangent vectors $\gamma_1'(0)$ and $\gamma_2'(0)$ generate the tangent plane in $f(u_0, v_0) = \gamma_1(0) = \gamma_2(0)$ — if there is a tangent plane (regular surface patch)

Normal vector $\vec{n}(u_0, v_0) \underset{\text{def}}{=} \gamma_1'(0) \times \gamma_2'(0)$

Implicit surfaces

Example: Sphere of radius r

1st parametric representation:

$$f\colon \begin{cases} [0,\, 2\pi)^2 \to \mathbb{R}^3 \\ (u,v) \mapsto \begin{pmatrix} r\cos u \cos v \\ r \sin u \cos v \\ r \sin u \end{pmatrix} \end{cases}$$

Polynomial representation $(u,v) \mapsto \begin{pmatrix} P(u,v) \\ Q(u,v) \\ R(u,v) \end{pmatrix}$?

$$P(u,v)^2 + Q(u,v)^2 + R(u,v)^2 = r^2 \implies P, Q, R \text{ constant}$$

Rational representation of the circle w/o $(r,\, 0)$**:**

$$\gamma\colon \mathbb{R} \to \mathbb{R}^2; \quad t \mapsto \left(r \cdot \frac{t^2 - 1}{t^2 + 1},\ r \cdot \frac{2t}{t^2 + 1} \right)$$

Rational representation of the sphere w/o $(r, 0, 0)$**:**

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3; \quad (u,v) \mapsto r \cdot \begin{pmatrix} \dfrac{(u^2-1)(v^2-1)}{(u^2+1)(v^2+1)} \\ \dfrac{2u(v^2-1)}{(u^2+1)(v^2+1)} \\ \dfrac{2u}{u^2+1} \end{pmatrix}$$

in homogeneous coordinates:

$$F\colon \mathbb{R}^2 \to \mathbb{P}^3; \quad (u,v) \mapsto r \cdot \begin{pmatrix} (u^2-1)(v^2-1) \\ 2u(v^2-1) \\ 2u(v^2+1) \\ (u^2+1)(v^2+1) \end{pmatrix}$$

Implicit representation of sphere:

$$x^2 + y^2 + z^2 = r^2$$

Normals and tangent planes to implicit surfaces

Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\}$ and $(x_0, y_0, z_0) \in S$.

To get the tangent plane. we have to linearize the equation:

$$\begin{aligned} & f(x_0 + ha, y_0 + hb, z_0 + hc) \\ = & f(x_0, y_0, z_0) + h \begin{pmatrix} a \\ b \\ c \end{pmatrix} \nabla f(x_0, y_0, z_0) + o(h) \\ = & h \begin{pmatrix} a \\ b \\ c \end{pmatrix} f(x_0, y_0, z_0) + o(h) \end{aligned}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ tangent vector} \Longleftrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \nabla f(x_0, y_0, z_0) = 0$$

The gradient of f is the normal vector to the surface

Example: Sphere $f(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0$

$$\nabla f(x_0, y_0, z_0) = 2 \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

The normal vector is parallel to the radius vector

CSG representation

CSG = computational solid geometry

Idee: Build a complicated scene using (few) primitives

Primitives

- **Solid modeling**

  simple polyhedrons like rectangular boxes, prisms, pyramids

  cones, spheres, tori, ...

- **Algebraic half spaces**

$$\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) \leq 0\}$$

- **Normed parts**

Regularized Boolean Operations

$$A \cup^* B \underset{\text{def}}{=} \overline{A^\circ \cup B^\circ}$$
$$A \cap^* B \underset{\text{def}}{=} \overline{A^\circ \cap B^\circ}$$
$$A \setminus^* B \underset{\text{def}}{=} \overline{A^\circ \setminus \overline{B}}$$

CSG trees

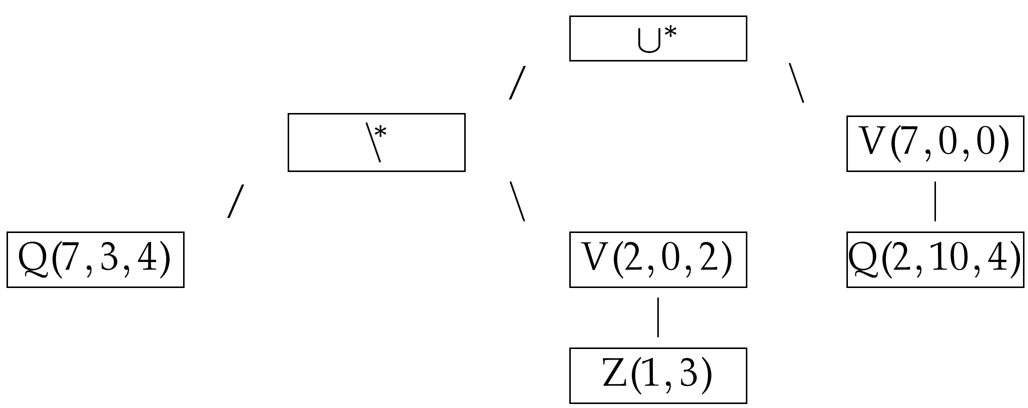
Let

$$Q(a,b,c) = [0,\, a] \times [0,\, b] \times [0,\, c]$$

and

$$Z(r,h) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\} \times [0,h]\,.$$

Let $T(a,b,c)$ denote translation by $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$



Classification of a point wrt a CSG object

The point can be **in**, **on** or **out**side of the object.

For primitives we can decide.

Propagate this from the leaves of the tree to the root.

$\cup^*$	*in*	*on*	*out*
in	*in*	*in*	*in*
on	*in*	**?**	*on*
out	*in*	*on*	*out*

$\cap^*$	*in*	*on*	*out*
in	*in*	*on*	*out*
on	*on*	**?**	*out*
out	*out*	*out*	*out*

$\setminus^*$	*in*	*on*	*out*
in	*out*	*on*	*in*
on	*out*	**?**	*on*
out	*out*	*out*	*out*

Curvature and distortions

Let γ be a curve patch.

$$s(t) = \int_a^t |\dot\gamma(t)| \quad \text{is monotone on } t$$

Natural parametrization: $\gamma(t) = \delta(s(t))$

Curvature: $\delta''(s)$

Circle:

$$\gamma\colon \begin{cases} [0, 2\pi) \to \mathbb{R}^2 \\ \quad t \mapsto (r\cos t, r\sin t) \end{cases} \qquad \gamma'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \end{pmatrix}$$

and $s(t) = rt$, hence

$$\delta\colon \begin{cases} [0, 2\pi r) \to \mathbb{R}^2 \\ \quad t \mapsto (r\cos \frac{t}{r}, r\sin \frac{t}{r}) \end{cases} \qquad \delta'(t) = \begin{pmatrix} -\sin \frac{t}{r} \\ \cos \frac{t}{r} \end{pmatrix}$$

Curvature vector $\delta''(t) = \dfrac{1}{r}\begin{pmatrix} -\cos \frac{t}{r} \\ -\sin \frac{t}{r} \end{pmatrix}$ has length $\dfrac{1}{r}$.

On surfaces: Take curvature vectors of curves on the surface.

Component in the tangent plane says nothing about the surface.

Normal curvature = normal component, depends only on the tangent vector.

Let κ_1, κ_2 be its minimum and maximum;
$\kappa = \kappa_1 \kappa_2$ is called GAUSSian curvature.

Theorema egregium: A mappings w/o distortions leaves κ invariant.