Algebraic and Semi-algebraic varieties

1. Affine and projective varieties

Let k be a field ($k = \mathbb{R}$, $\mathbb{C}$ or computable subfield thereof)

Let $f_1, \ldots, f_r$ be polynomials in $X_1, \ldots, X_n$.

$$V(f_1, \ldots f_r) = \{(x_1, \ldots, x_n) \in k^n \mid f_i(x_1, \ldots, x_n) = 0 \quad \forall i = 1 \ldots r\}$$

is the affine variety defined by $f_1, \ldots, f_r$.

What about points with homogeneous coordinates?

$(x : y : z : u)$ and $(\lambda x : \lambda y : \lambda z : \lambda u)$ are the same point for any $\lambda \neq 0$.

A polynomial F is called *homogeneous* of degree d, if

$$F(\lambda x, \lambda y, \lambda z, \lambda u) = \lambda^d F(x, y, z, u) \quad \forall \lambda \in k$$

Then, for $\lambda \neq 0$ we have $F(x, y, z, u) \neq 0 \iff F(\lambda x, \lambda y, \lambda z, \lambda u) \neq 0$

Let $\mathbb{P}^n(k)$ denote the set of all points with $(n+1)$ homogeneous coordinates in k, i.e. a point of $\mathbb{P}^n$ can be written as

$$(x_1 : \ldots : x_{n+1})$$

with at least one $x_j \neq 0$.

For hogeneous polynomials $F_1, \ldots, F_r$ in $X_1, \ldots, X_{n+1}$, we call

$$V(F_1, \ldots F_r) = \{(x_1 : \ldots : x_n) \in \mathbb{P}^n(k) \mid F_i(x_1, \ldots, x_{n+1}) = 0 \quad \forall i = 1 \ldots r\}$$

the projective variety defined by $F_1, \ldots, F_r$.

Example: Circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

As a projective variety:

$$\{(x : y : z) \in \mathbb{P}^2(\mathbb{R}) \mid x^2 + y^2 = z^2\}$$

Hyperbola: $\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}$

As a projective variety:

$$\{(x : y : z) \in \mathbb{P}^2(\mathbb{R}) \mid x^2 - y^2 = z^2\} = \{(x : y : z) \in \mathbb{P}^2(\mathbb{R}) \mid y^2 + z^2 = x^2\}$$

Projectively, **Hyperbola = Circle !**

$$\begin{aligned} &\{(x : y : z) \in \mathbb{P}^2(\mathbb{R}) \mid x^2 - y^2 = z^2\} \\ = &\{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid x^2 - y^2 = 1^2\} \cup \{(x : y : 0) \in \mathbb{P}^2(\mathbb{R}) \mid x^2 - y^2 = 0^2\} \\ = &\{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid x^2 - y^2 = 1^2\} \cup \{(1 : 1 : 0), (1 : -1 : 0)\} \end{aligned}$$

The two points at infinity close the hyperbola.

Applications in Modelling and Visualization

- Implicit surfaces are algebraic varieties
- Patches of such surfaces can be use to modell or approximate arbitrarily complicated surfaces (A-*patches*, BAJAJ)
- Boundaries of algebraic halfspaces are varieties
- …

Semialgebraic varieties

Only definied over the real numbers (or a formally real field)

A semialgebraic variety is a finite union of sets of the form

$$\{(x_1,\ldots,x_n)\in\mathbb{R}^n \mid f_i(x_1,\ldots,x_n)\ R_i\ 0 \quad \forall i=1\ldots r\}$$

where $f_1,\ldots,f_r$ are real polynomials in $X_1,\ldots,X_n$
$R_1,\ldots,R_r \in \{=,\neq,<,>,\leq,\geq\}$

Example: Every algebraic half space $\{(x,y,z)\in\mathbb{R}^3 \mid f(x,y,z)\leq 0\}$ is a semialgebraic variety.

Intersections of semialgebraic varieties are semialgebraic:
Combine the relations

Unions of semialgebraic varieties are semialgebraic by definition

Complements of semialgebraic varieties are semialgebraic:
If there is only one relation, take its negation.
If there are more, take the union of the complements of the semialgebraic varieties defined by the individual relations.

The closure of a semialgebraic variety is semialgebraic:
Replace all $<$ and $>$ by $\leq$ and $\geq$, discard all $\neq$-relations.

The interior of a semialgebraic variety is semialgebraic:
Replace all $\leq$ and $\geq$ by $<$ and $>$, discard all equations.

Every CSG-object built from algebraic half spaces is semialgebraic.

Cylindrical decomposition

Theorem of TARSKI and SEIDENBERG *(special form)*

If $Z \subset \mathbb{R}^n$ is semialgebraic, so is its image under any projection from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$.

NB: This is **not** true for algebraic varieties: The projection of a circle from $\mathbb{R}^2$ to $\mathbb{R}^1$ is an intervall, which is not an algebraic variety!

Idea (COLLINS): Describe a semialgebraic variety in $\mathbb{R}^n$ recursively in terms of its projection to $\mathbb{R}^{n-1}$ under the obvious projection

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$$

Starting points: Semialgebraic varieties in $\mathbb{R}$

$\{x \in \mathbb{R} \mid f(x) = 0\}$ is a (possibly empty) finite set of points.

$\{x \in \mathbb{R} \mid f(x) = 0\}$ is $\mathbb{R}$ minus a (possibly empty) finite set of points.

$\{x \in \mathbb{R} \mid f(x) < 0\}$ is a union of finitely many (possibly infinity) open intervalls.

$\{x \in \mathbb{R} \mid f(x) > 0\}$ is a union of finitely many (possibly infinity) open intervalls.

$\{x \in \mathbb{R} \mid f(x) \leq 0\} = \{x \in \mathbb{R} \mid f(x) < 0\} \cup \{x \in \mathbb{R} \mid f(x) = 0\}$

$\{x \in \mathbb{R} \mid f(x) \geq 0\} = \{x \in \mathbb{R} \mid f(x) > 0\} \cup \{x \in \mathbb{R} \mid f(x) = 0\}$

Any semialgebraic variety in $\mathbb{R}$ is a union of finitely many (possibly infinity) open intervalls and points.

Structure theorem

If $Z \subset \mathbb{R}^n$ is semialgebraic and $\pi: \mathbb{R}^n \to \mathbb{R}^{n-1}$ the usual projection, there exists an effectively computable decomposition

$$\pi(Z) = W_1 \cup W_2 \cup \cdots \cup W_r$$

into disjoint semialgebraic varieties $W_i \subset \mathbb{R}^{n-1}$, such that for each W_i, the cylinder $W_i \times \mathbb{R}$ can be decomposed into (computable) semialgebraic sets W_{ij}, each of which can be written in one of the following forms:

a) Sets of *band* type:

$$W_{ij} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{c} h(x_1, \ldots, x_{n-1}) \in W_i \quad \text{and} \\ f(x_1, \ldots, x_{n-1}) \leq x_n \leq g(x_1, \ldots, x_{n-1}) \end{array} \right\}$$

b) Sets of *graph* type:

$$W_{ij} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1, \ldots, x_{n-1}) \in W_i \wedge x_n = f(x_1, \ldots, x_{n-1})\}$$

f and g are continous functions.

Applications: See where the points of Z are located
Piano mover's problem,
Motion planning for robots,
...

Invariants and classification

When are two polygones geometrically the same?

1st answer: If there is a EUCLIDIAN transformation moving one onto the other.
2nd answer: It all corresponding angles and all lenghts of corresponding edges are the same.

Why are both answers equivalent?

EUCLIDIAN transformation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where A is an orthogonal matrix.

Look for functions depending on n points $P_i = (x_i, y_i, z_i)$ invariant under all those transformations.

Invariance under translation implies: The function should only depend on difference Vectors $\overrightarrow{P_i P_j}$.

Orthogonal matrices are characterized by the fact that they respect scalar products between vecors or, equivalently, lengths of vectors and angles between vectors.

FELIX KLEIN, **Erlangener Programm** (1872):

Every geometry studies the invariants of a group:

EUCLIDian geometry — group of EUCLIDian motions
Affine geometry — group of affine maps
Projective geometry — group of projective maps, i.e. maps described by 4×4-matrices acting on homogeneous coordinates

Application: How to decide whether two objects in space are really the same?
Compute their invariants!

When are two algebraic varieties really the same?

$V \subset \mathbb{P}^n(k)$ and $W \subset \mathbb{P}^m(k)$ are called *isomorphic*, if there is a bijective map $\varphi: V \to W$ such that both φ and φ^{-1} can be defined by polynomials.

To check if V and W are isomorphic, embedd them suitably into a common projective space and compute their invariants.

To understand *all* projective varieties with given properties (like dimension, number of connected components, . . .),
first embed them all into a common projective space,
then look for invariants.

Curves and function fields

Let $V \subset k^n$ be an affine variety and let

$$I(V) = \{f \in k[X_1, \dots, X_n] \mid f(V) = 0\}.$$

Two polynomials $g, h \in k[X_1, \dots, X_n]$ coincide as functions restricted to V, iff $g - h \in I(V)$. The set

$$\mathcal{O}(V) = k[X_1, \dots, X_n]/I(V)$$

of all functions on V is called the *coordinate ring* of V.

If it is a domain, i.e.

$$g \times h = 0 \Longrightarrow g = 0 \vee h = 0,$$

we can take the quotient field

$$k(V) = \left\{ \frac{g}{h} \;\middle|\; g \in \mathcal{O}(V),\ h \in \mathcal{O}(V) \setminus \{0\} \right\} /_{\sim}$$

where

$$\frac{g}{h} \sim \frac{k}{\ell} \Longleftrightarrow g\ell = hk.$$

V is then called irreducible, and $k(V)$ is the function field of V.

V is called a curve, if there exists a $t \in k(V)$, such that every $f \in k(V)$ depends algebraically on t.

Theorem: Isomorphic curves have isomorphic function fields, and a nonsingular curve is determined uniquely upto isomorphism by its function field.

The second part is no longer true in higher dimensions!

3D reconstruction of algebraic curves

Let C be a curve in 3-space and consider to projections C_1, C_2 of C into planes with centers O_1 and O_2.

Usually, the projections $C \to C_i$ will be at least generically one-to-one; only for special positions of the center worse things will happen.

If the projections are one-to-one, $k(C) = k(C_1) = k(C_2)$; hence C is determined upto isomorphism (i.e. as an abstract curve) by C_1 and by C_2.

In fact, generically a sufficently complicated curve C is even determined upto a projective transformation of $\mathbb{P}^3(k)$, i.e. as a space curve.

Idea of proof: An embedding $C \hookrightarrow \mathbb{P}^3(k)$ is given by a threedimensional sub vector space W of $k(C)$.

The two embeddings $C_i \hookrightarrow \mathbb{P}^2(k)$ are given by two dimensional sub vector spaces $W_i \leq W$.

Only special curves have many automorphism; most general curves have none. Then the intersection $W_1 \cap W_2$ can be found both in W_1 and in W_2, and thus W can be constructed.